Risk-Based Regulations in Credit Markets: A Heterogeneous Risk Accelerator

Zhiyu Fu*

Abstract

Credit markets in the U.S. are dominated by institutional investors, whose risk capacity is limited by various risk-based regulations. I study the macroeconomic implications of such risk-based regulations in a general-equilibrium model featuring firms with heterogeneous credit risks and a bond investor subject to risk-based constraints. During economic downturns, these risk-based constraints become a *heterogeneous risk accelerator*: It increases the debt financing cost for risky firms, amplifying their default risk, while generating convenience yields for the safest firms. In aggregate, these constraints significantly amplify the drop in investment and output. I evaluate the effects of credit market intervention programs using this framework. I find that during credit market disruptions, credit facilities mitigate the initial damage and speed up the follow-up recovery.

1 Introduction

Credit markets in the U.S. are dominated by financial institutions, such as banks in the loan market and insurance companies and mutual funds in the corporate bond market. These institutions are subject to various regulations in their portfolio allocation. Such regulations are often risk-based—financial institutions are often required to hold more capital against assets with higher risks. For example, the Risk-Based Capital (RBC) regulation on insurance companies requires a higher equity buffer for assets with higher credit risks (Becker and Ivashina, 2015), and the Basel Accords impose similar requirement on banks (BCBS, 2011). Such risk-based regulations naturally lead to differential effects on bonds with different credit risks, and hence also heterogeneous effects on the real outcomes of issuing firms. Nonetheless, despite the

*I thank Fernando Alvarez, Manav Chaudhary, Lars Peter Hansen, Zhiguo He, Greg Kaplan, Ralph Koijen, Jingoo Kwon, Jane Li, Ben Marrow, Robert Shimer, Quentin Vandeweyer, and seminar participants at Capital Theory, Econ Dynamics and ASSA 2023 poster sessions, for their insightful comments and suggestions. All errors are my own. *Email:* zhiyufu@uchicago.edu

prevalence of risk-based regulations in credit markets and the crucial role of credit markets in the real economy, their macroeconomic implication has been rarely explored.

The goal of this paper is to provide a quantitative framework to study how risk-based regulations on financial intermediaries affect the real economy, and suitable policy interventions to counteract if needed. To this end, I build a general-equilibrium model with a sector of of heterogeneous firms and a representative institutional investor. Firms choose optimal investment and long-term bond issuance dynamically. They are subject to idiosyncratic capital quality shocks, which induce diffusion in leverage and hence heterogeneous credit risks. Institutional investors hold corporate bonds but are constrained by risk-based regulations as in the real world. Using this model, I study the effect of risk-based regulations on investment and output after a crisis shock that mimics the Covid-19 recession in 2020. I also evaluate the policy consequences of the credit market interventions by the Federal Reserve.

I propose a novel mechanism caused by risk-based regulations, called *heterogeneous risk accelerator*. It has the classic financial accelerator feature (Kiyotaki and Moore, 1997; Bernanke et al., 1999; Brunnermeier and Sannikov, 2014): during economic downturns, firms' credit risks endogenously increase, leading to higher portfolio risks for bond investors and thereby binding risk constraints. To comply with regulations, bond investors are willing to take risky bonds only at discounts. Therefore, risky firms now face higher financing costs in order to roll over their debt. They borrow more and also cut off investment to keep themselves afloat, amplifying the default risk. The higher default risk further feeds back into risk constraints, triggering another round of amplification.

The novelty of this mechanism lies in its redistributional effects: Along the spectrum of credit risk, risk-based regulations financially suppress risky firms while subsidizing the safest firms. Firms with higher credit risks are more severely hurt by the accelerator, as their bonds are assigned higher risk weights in the portfolio and receive higher discounts. In contrast, safe firms actually benefit from such regulations. Their bonds now carry a "convenience premium", granting them preferential access to the credit market. The convenience premium comes from the general-equilibrium effect: with the same aggregate demand for savings, when savings in risky bonds are constrained, the extra demand is poured into the less-constrained safe bonds, pushing up its price. Whether this redistribution reduces or exacerbates the drop in aggregate investment depends on how firms are distributed on the spectrum of credit risks. In real world, most firms are rated as BBB, which carries significant risk weights and is penalized by regulations. This salient feature of data is reflected in calibration. Therefore, in aggregate, risk-based regulations aggravate the drop in investment and outputs.

In this model, credit market intervention programs are effective at stimulating the economy during a crisis. The stimulus program is modeled as government purchasing corporate bonds financed by issuing government liabilities, as do those corporate credit facilities (CCFs) established by the Fed in March 2020.[1] The stimulus program relieves risk constraints by taking risky assets off financial institutions' balance sheets. In my numerical experiments, I find such credit facilities successfully speed up the recovery of the economy from a credit market meltdown. It also captures the announcement effect of such policies: upon the announcement of the program, yields in the credit market drop immediately even before the purchase begins. This is because the announcement changes the market belief of default risks in the near future, and hence reduces the risk weights immediately. This announcement effect is consistent with empirical findings using high-frequency approaches around FOMC meetings (Gilchrist et al., 2020).

Below I provide an overview of my method and quantitative results. I start in Section 2 with a brief review of risk-based regulations for banks, insurance companies, and mutual funds. Banks face risk-weighted capital requirements in the Basel Accords, insurance companies are subject to risk-based capital (RBC) requirements imposed by the National Association of Insurance Commissioners (NAIC), and bond mutual funds often have restrictions on the share of risky assets either for their internal risk management or by their investment mandates. These regulations usually involve a risk metric, which maps assets into risk scores, and a hard cap on the total risk scores in the portfolio. The risk metric may vary across industries and institutions. For example, large banks often have an internal model to estimate the credit risk of an asset and a risk-weight function to convert the credit risk to risk weights. One widely used metric in credit markets is the credit rating by credit rating agencies such as Moody's and S&P. Credit ratings are discrete, creating discontinuities, particularly around the cutoff of investment-grade and high-yield bonds. The discrete feature creates market segmentation that are well-documented in the literature (Ellul et al., 2011; Chernenko and Sunderam, 2012; Becker and Ivashina, 2015). These realistic features of risk regulations are incorporated in my model framework.

Section 3 describes my framework. The model has three types of market participants in the economy: a representative household, who is the ultimate owner of all securities in the economy, a sector of heterogeneous firms, financed by equity and long-term defaultable bonds, and a financial intermediary that invests in bonds on behalf of the household. There is also a government in the model, issuing government bonds and collecting taxes from firms to pay interests. For trackability, this model does not feature anticipated aggregate risks.[2] Instead, I calibrate the model at the steady state, and hit the economy with a probability-zero crisis shock

[1]In real world, the Fed purchases corporate bonds by releasing reserves into the financial system. In a real model without nominal rigidity, it is equivalent to issuing government debt.

[2]With aggregate risks and heterogeneous firms, the whole distribution of firms becomes state variables of the model, dramatically increasing the computational complexity. I follow the common practice in the heterogeneous-agent literature by assuming away aggregate risks and focusing on risks induced by idiosyncratic shocks. Without the aggregate risk, the economy evolves deterministically and therefore aggregate states can be summarized using the time dimension alone. Recently computational methods have also been also developed in the literature to overcome the curse of dimensionality, such as Schaab (2020) using adaptive sparse grid and Fernandez-Villaverde et al. (2019) using deep learning.

(an "MIT" shock) to study the transition dynamics in crisis.

Section 3.2 describes firms' behavior. Firms face frictions in equity issuance, and finance their investment by borrowing on the credit market via long-term bonds. Firms are identical at birth, but the different realizations of idiosyncratic capital quality shocks lead to heterogeneous leverage ex-post. Leverage gives rise to credit risks: when a firm's leverage is above a certain threshold after a series of negative shocks, it is unable to roll over its debt and has to enter the bankruptcy process, liquidating its capital to pay off debt holders.

Section 3.3 describes the pricing of bonds by financial intermediary. The price of a corporate bond equals to the future cash flow discounted at suitable discount rates. As all idiosyncratic shocks are diversified away in their portfolio, absent of frictions, the discount rates for all bonds are equalized to the risk-free rate by no-arbitrage. However, just as in the real world, the intermediary is subject to two risk constraints. A risk-weight (RW) constraint assigns each bond a risk weight and limits the total risk weights in the portfolio. The risk weight is continuously increasing in the credit risk of a bond. Another high-yield (HY) constraint limits the share of portfolio in high-yield bonds, i.e., bonds whose default probabilities are higher than a given threshold. These two constraints allow for both sensitive responses in risk weights to credit risks as well as potential market segmentation between investment-grade (IG) and high-yield (HY) bonds. When constraints bind, bonds with higher risk weights (or in the high-yield segment) are discounted at higher rates, while in contrast the government bond and the safest corporate bonds appreciate as a "safe haven".

The equilibrium between the pricing of long-term bonds and firms' optimal policies is a rather complicated object and worth a short discussion here. The challenge is that it is a high-dimensional equilibrium, as firms need to know bond prices *at each possible state* in the future in order to make the optimal decision, while bond investors also need to know firms' optimal policies *at each state* to estimate the default risk. In the language of the model, this model gives a coupled system of partial differential equations (PDEs) with optimal controls. This setup closely follows the corporate finance literature on dynamic leveraging, represented by Demarzo and He (2020). Their model features coupled ODEs, but they manage to decouple them thanks to the analytical tractability of their model. My model features much richer frictions, so the coupled PDEs have to be solved numerically. I provide an algorithm that can solve it efficiently and nest it into a general equilibrium. This numerical algorithm opens doors for such models to many broad quantitative applications.[3]

I calibrate the model in Section 4. Crucial to my quantitative results are the distribution of firms along the credit risk spectrum and the risk-weight function. My model captures the empirical stylized fact that default is generally a rare event (around 0.5% per year, dollar-weighted). By

[3]Similar models have been built in discrete time, such as Kuehn and Schmid (2014); Gomes et al. (2016); Gomes and Schmid (2020).

calibration, high-yield firms only account for 15% in the corporate bond market in the model, and most firms carry medium levels of credit risks, equivalent to firms rated as BBB in real world. I calibrate a continuous risk-weight function using *the standardized approach* from the Basel Accords, which assigns zero weight to the U.S. government bond, and is very sensitive to small to medium risks. Figure 1 summarizes key features of my calibration.

I then introduce an aggregate transitory shock to the economy in Section 5.1 and study the transition dynamics with and without constraints. The shock is motivated by the COVID-19 recession in the U.S. in 2020, featuring a 10% drop in aggregate productivity and a 20% increase in firms' volatility. I consider three different scenarios after the shock: an unconstrained scenario, a constrained scenario where only the risk-weight constraint is active (RW-constrained), and a scenario where both constraints are active (RW&HY-constrained). The recession under the unconstrained scenario is mild: Default probabilities and yields increase slightly, reflecting lower-than-usual short-term fundamentals, and aggregate investment drops by around 10% but recovers quickly.

In the RW-constrained scenario, due to higher default risks and therefore larger risk weights, the risk-weight constraint endogenously binds, raising corporate bond yields. Higher yields increase financing costs for firms, hampering investment and accelerating default. Investment drop by 70% for an average high-yield firm, and 25% for an average investment-grade firm. On the contrary, despite the negative TFP shocks, the safest firms do not reduce their investment at all, thanks to the convenience premium carried in their bonds. In aggregate, investment drops by 15% more relative to the unconstrained scenario.

The role for the additional HY constraint is mostly redistributional: It further suppresses prices of HY bonds, while relieving financing costs for IG firms. Its aggregate effect is small, and if anything, the aggregate investment is even slightly higher under the RW&HY-constrained scenario. This is because high-yield firms are subject to strong *debt-overhang:* As they are closer to default, the benefit of additional investment will be mostly absorbed by debt holders. This reduces equity holders' incentive to invest. By redistributing credits away from such firms to healthy firms, the HY constraint actually improves aggregate investment. The effect is small in aggregate because by calibration, high-yield firms account for a small share of the market.

In Section 5.2, I study the effect of credit market intervention programs in response to the crisis shock. As in the real world, the government issues government debt (bank reserves) to purchase investment-grade corporate bonds. Corporate bond yields and default probabilities drop upon the announcement even before the purchase begins, as bond investors rationally expect that the coming stimulus can relax their binding risk constraints in the near future. Lower yields immediately boost investment. As the government gradually increases the purchase, bond yields further decrease back to pre-crisis levels. Overall, the stimulus policy greatly speeds up the recovery relative to laissez-faire. I also compare this policy with an alternative policy which only purchases high-yield bonds, in the hope that it can relax the constraint more efficiently. Mirroring the results on the HY constraint, my quantitative result shows that the gain from an HY-targeting policy is negligible in aggregates.

Here I conclude the introduction with a short discussion on the policy implications. Admittedly, this model is not suitable for evaluating the trade-offs of risk-based regulations. It is because the regulations are not justified within the model: in real world, risk-based regulations are imposed to limit the risks of financial institutions, while in this model, financial institutions do not bear risks due to diversification. Instead, I take the existence of such regulations as a primitive, and study its quantitative effect on shock responses. In this sense, the results should be interpreted positively rather than normatively. For this purpose, introducing aggregate risks to this model is unlikely to overturn the major results. If anything, the mechanism studied here will be amplified, as the fear of a potentially binding constraint in the future will disproportionately hurt risky firms. I argue such evaluation is still helpful for understanding the functions of these regulations, and therefore can better inform future policies.

Literature Review and Contribution. This paper contributes to multiple strands of the literature. First, it builds on the classic financial accelerator literature that studies the role of financial frictions in asset prices and the macroeconomy. This literature is pioneered by Bernanke and Gertler (1989), Kiyotaki and Moore (1997), and synthesized in Bernanke et al. (1999). In their model, entrepreneurs face a collateral borrowing constraint. A small negative shock can trigger adverse feedback loops due to a binding constraint, resulting in amplification and propagation. The Great Recession in 2008 revived this literature, shifting researchers' attention to financial frictions. Brunnermeier and Sannikov (2014) cast the model in continuous time and solve the model with full uncertainty, as opposed to the log-linearization technique used by the earlier literature. In light of the financial crisis, Gertler and Kiyotaki (2010); Gertler and Karadi (2011) build a workhorse model to emphasize the role of financial intermediaries (banks) and to study the effect of credit policies. Recently several studies also directly model capital requirement for banks, and study its normative and positive implications for macroeconomy (Begenau, 2020; Pancost and Robatto, 2023). The contribution of this paper to the literature is twofold. First, I study *risk-based* regulations in the credit market. Despite its prevalence, their role in the macroeconomy is rarely studied. Most models in this literature treat such regulations as a uniform leverage constraint, neglecting their risk-based feature. [4] Second and more importantly, I show that effects of the financial accelerator are highly heterogeneous, with risky firms being punished and safe firms being subsidized.

My work also speaks to the empirical literature on risk-based regulations in the credit market.

[4] A notable exception is Repullo and Suarez (2013), who study the procyclicality of bank capital regulations due to default risks in a dynamic equilibrium model.

On the bank loan side, Behn et al. (2021) show that banks whose regulation is more sensitive to credit risks (banks who adopt the internal-ratings-based approach, see discussion in Section 2) cut off loans more severely in response to negative shocks than other banks do. On the corporate bond side, Ellul et al. (2011) empirically show the impact of rating-based regulations on bond prices. They find that bonds subject to a higher probability of regulatory-induced selling exhibit larger price declines and subsequent reversals. Chernenko and Sunderam (2012) show that the market segmentation in the corporate bond market has real effects. They observe that flows into high-yield mutual funds have an economically significant effect on the investment of high-yield firms relative to investment-grade firms near the cutoff. This paper, inspired by these empirical findings, build a structural model featuring such regulations to understand its quantitative role in a general equilibrium, which is often difficult to study in reduced-form analyses.

This paper also contributes to the burgeoning literature on the effects of the unprecedented corporate bond purchases by the Federal Reserve in response to the COVID-19 shock. Empirically, Haddad et al. (2021); Nozawa and Qiu (2021); Gilchrist et al. (2020), among others, show that the Fed's announcement of corporate bond purchases successfully stabilized the credit market. These studies usually employ a high-frequency approach to detect the responses in the financial market. However, empirically detecting the real effects of such policies is challenging, in particular amid market turmoil. This paper takes a quantitative structural approach to tackle this question, along with others such as Brunnermeier and Krishnamurthy (2020); Crouzet and Tourre (2021); Chang (2022). Crouzet and Tourre (2021) is the closest to this paper in terms of modeling techniques. They argue that the stimulus policy can be effective during financial market disruptions. The current paper echoes this view with a particular source of financial disruption in mind: binding risk constraints.

2 Institutional Background

In this section I briefly discuss the institutional background on risk-based regulations that motivates my modeling choices. Corporate debt is commonly issued in the form of corporate bonds or loans. Investors in these markets are often large financial institutions—the corporate bond market is dominated by insurance companies and mutual funds, while the loan market is dominated by banks. Though those financial institutions differ greatly in many aspects, they all face some forms of risk-based regulations that limit credit risks they can take in their investment.

The regulation on banks is in the form of capital requirement: banks need to maintain a minimum level of capital (equity) subject to certain rules. In the case when their loans do not pay off, losses can hopefully be absorbed by the equity buffer, so banks still have sufficient assets to pay off depositors. In the early years, the capital requirement simply sets a minimum level of capital to deposit ratio, but does not differentiate between assets with different levels of risk. Therefore, a bank that lends mostly to AAA companies is considered as "equally strong" in this approach as a bank whose clientele are mostly junk bond issuers, as long as their capital-deposit ratios are the same. This rule fails to curb the risk-shifting tendency of the under-capitalized banks: as predicted by the classic corporate finance theories (Jensen and Meckling, 1976), banks who are short on capital will be inclined to take more on risks, effectively shifting risks to its depositors.[5]

To overcome this drawback, the first Basel Accord (Basel I) introduced *risk-weighted assets* in 1988. In essence, the new regulation requires a bank to hold more capital for riskier assets. Basel I groups banks' assets into five risk categories based on asset classes, and assign risk weights accordingly. The U.S. government debt has zero risk weight, while private sector debt is all assigned with the highest risk weight, regardless of its credit risks.

The insensitivity of risk weights to credit risks in Basel I had been widely criticized. In 2004, the Basel Committee introduces more granularity into the second Basel Accord (Basel II). In Basel II, two approaches for risk-weighting are allowed: the standardized approach, and the internal ratings-based (IRB) approach. Under the standardized approach, the risk weights on corporate claims are assigned based on external credit ratings. Under the IRB approach, banks are allowed to use their internal models to estimate risk parameters (e.g., the probability of default), and translate them into risk weights using *risk-weight function*s. One advantage of the IRB approach is that internally estimated risk weights are more sensitive to credit risks than discrete credit ratings.[6] The risk-weight methodology remains mostly unchanged in Basel III.

The Risk-Based Capital (RBC) requirement for insurance companies has been going through similar developments as those for banks. The RBC requirement is prescribed by the National Association of Insurance Commissioners (NAIC), and has been implemented at the state level since the 1990s. The purpose of the RBC requirement is to identify weakly capitalized companies to ensure that policyholders will receive the benefits promised by the insurance companies. Similar to the standardized approach in Basel Accords , the RBC requirement for insurance companies also assigns a zero risk weight to the U.S. government bonds, and assign risk weights to corporate bonds according to their credit ratings. Before 2019, there were only six risk designations in total, so bonds with meaningfully different risks (e.g., AAA vs A) may be assigned with the same risk weights. This relatively crude risk-weight function leads to insurers' bond portfolio concentrated in the risky end of each designation (the so-called "reaching-for-yield" phenomenon, documented by Becker and Ivashina, 2015). It also creates mechanical cliffs between bonds with similar risks but different risk designations. In 2019, the NAIC revised the risk-weighting methodology to add more granularity to the risk-weight function. After

[5]See Haubrich (2020) for a brief overview on the history of bank capital requirements in the U.S.

[6]Researchers have also criticized the IRB approach as banks can manipulate their internal risk-weighting models to reduce effective capital requirements. See Vallascas and Hagendorff (2013); Mariathasan and Merrouche (2014).

this update, there are 20 designation categories based on credit ratings, allowing for more continuous changes in risk weights to underlying risks. The current risk-weighting method is very similar to the standardized approach of Basel II. [7] Nevertheless, cliffs between designations still exist, especially between the investment-grade and high-yield cutoff: downgrading from BBB (investment-grade) to BB (high-yield) almost doubles the capital requirement for that security. In addition to the RBC requirement, the NAIC also prescribes another hard cap of 20% for all non-investment-grade bonds in the portfolio of insurance companies. This cap is generally not binding (Ellul et al., 2011).

For bond mutual funds, there is no regulatory restriction imposed by the government that directly limits their risk-taking. Nevertheless, they are still subject to various risk-based constraints in their portfolio. As an industry standard, mutual funds actively manage their portfolio risks by setting a risk limit, using risk metrics such as Value-at-Risk (VaR). This practice is often referred to as "risk budgeting" (Pearson, 2011). Asset allocations under the risk constraint often deviate from the optimal policy of a utility-maximizing investor (Basak and Shapiro, 2001). Furthermore, bond mutual funds often specialize in either investment-grade (IG) or high-yield (HY) bonds. Mutual funds specializing in the investment grade usually have rating-based investment mandates that require portfolio managers to hold minimum shares of portfolios in the investment grade. For example, PIMCO total return fund states in the mandate that "the Fund invests primarily in investment-grade debt securities, but may invest up to 20% of its total assets in high yield securities". Some funds may also seek to track corporate bond indices, many of which exclusively consist of investment-grade bonds. These funds will also avoid investing in high-yield bonds to minimize tracking errors.

Several common features of risk regulations can be found from the discussion above. Risk regulations are often imposed to mitigate the excessive risk-taking that could arise from a typical principal-agent problem. Risk regulations are often in the form of risk limits in the portfolio measured by certain risk metrics. These risk metrics, in theory, should sensitively reflect credit risks of assets, while in practice it sometimes relies on discrete credit ratings, creating discontinuity and market segmentation. These features of risk regulations are reflected in my model, discussed in 3.3.

3 Model

Time is continuous. There are three types of market participants in the economy: a representative household, who is the ultimate owner of all securities in the economy, a sector of heterogeneous firms, financed by equity and long-term defaultable bonds, and a representative

[7] See Figure 6 in Obersteadt (2017).

financial intermediary that invests in bonds on behalf of the household. The financial intermediary is subject to regulatory constraints detailed in Section 3.3. In addition to these market participants, a government taxes firms' after-interest earnings and redistributes to the household. There is no anticipated aggregate risk, so the aggregate economy evolves deterministically. Later, I will hit the economy with a probability-zero aggregate shock (i.e., an MIT shock), and study the transition dynamics.

3.1 Household

We start with the representative household, the ultimate owner of all securities in financial markets. The representative household maximizes her discounted utility by choosing consumption plan C_t, labor supply L_t, and portfolio allocation between firm equity (W_t^E) and the bond intermediary (W_t^B). As idiosyncratic risks are fully diversified away at the portfolio level, returns from both sources are risk-free. Furthermore, as the household can freely adjust capital allocations, instantaneous returns from both sources have to be equalized, denoted as r_t^f. The optimization problem of the household therefore can be written as

$$\begin{aligned}
&\max_{C_t, L_t} \int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \beta \frac{L_t^{1+\nu}}{1+\nu} \right) dt \\
&s.t. \\
&dW_t = \left(r_t^f W_t + w_t L_t + T_t - C_t \right) dt,
\end{aligned} \tag{3.1}$$

where T_t is the lump-sum transfer from the government. The solution to the household's maximization problem gives the classic Euler equation and the labor supply function. The former links the risk-free rate to consumption, and the latter links consumption to labor supply:

$$\frac{dC_t}{C_t} = \frac{1}{\gamma} \left(r_t^f - \rho \right) dt \tag{3.2}$$

$$\beta C_t^{\gamma} L_t^{\nu} = w_t \tag{3.3}$$

3.2 Firms

The economy is populated by a continuum of firms indexed by their capital K_t^i and debt in face value B_t^i. Firms are subject to idiosyncratic capital quality shocks that are i.i.d. both across time and firms. As idiosyncratic shocks are fully diversified in aggregate by the law of large number, firms' distribution evolves deterministically. At each instant, firms produce, (dis-)invest, service their debt, issue/repurchase debt, and pay out the remaining earnings as dividends. Investment and debt issuance are chosen optimally by firm managers to maximize expected values of dividends, discounted at the risk-free rate r_t.

Production. In every instant, each firm produces $Y_t^i dt$ according to a Cobb-Douglas production function $Y_t^i = Z_t \left(L_t^i\right)^{\alpha} \left(K_t^i\right)^{1-\alpha}$, where L_t^i is the quantity of labor hired from a competitive labor market. Firms can adjust labor freely, so they simply choose labor L_t^i to maximize instantaneous productions net of labor expenses, $Y_t^i - w_t L_t^i$, taking wage w_t as given. This leads to a standard labor demand function linear in capital:

$$L_t^D(K) = \left(\frac{\alpha Z_t}{w_t}\right)^{\frac{1}{1-\alpha}} K. \tag{3.4}$$

Define $\tilde{Z}_t \equiv (1-\alpha)\left(\frac{\alpha Z_t}{w_t}\right)^{\frac{\alpha}{1-\alpha}} Z_t$ as the effective productivity per unit of capital. Firms' earnings before interest, taxes, and depreciation (EBITDA), can be written as $Y_t^i - w_t L_t^i \equiv \tilde{Z}_t K_t^i$. Two features arise from this formulation: Firms' earnings are linear in capital; all firms face the same effective productivity $\tilde{Z}_t$ since they are competing in the same labor market and share the same aggregate productivity. Therefore, in describing firms' behavior below, I proceed as if $\tilde{Z}_t$ is exogenous and firms have a linear technology.

Investment. Firms face idiosyncratic capital quality shocks that are proportional to their capital stock. Their capital evolves according to the law of motion:

$$dK_t^i = \left(I_t^i - \delta K_t^i\right) dt + \sigma K_t^i d\mathcal{B}_t^i, \tag{3.5}$$

where I_t^i is the investment optimally chosen by firm i, δ is the depreciation rate, which is tax-deductible, and $d\mathcal{B}_t^i$ is the increment of a standard Brownian motion, independent from each other. When the firm actively adjusts their capital stock via investment, they also need to pay a capital adjustment cost $\varphi^K(\frac{I_t^i}{K_t^i})K_t^i dt$, where $\varphi^K(\iota) \equiv \frac{\varphi_k}{2}(\iota-\delta)^2$ is convex and increasing the investment rate ι.

Financing. To finance investment, firms can issue corporate debt. Following the standard assumption in the long-term debt literature,[8] I assume that debt takes the form of exponentially-maturing coupon bonds. Denote B_t^i as the face value of outstanding bonds issued by firm i. One unit of bond pays a constant coupon rate $c > 0$, so that over $[t, t+dt]$, debt holders receive coupon payment of $cB_t^i dt$ from firm i in total. The coupon payment is also tax-deductible, which shields $\tau c B_t^i dt$ in corporate taxes. This tax shield benefit of debt provides the reason for

[8] e.g. Demarzo and He (2020); Kuehn and Schmid (2014); Philippon (2009).

unleveraged firms to take leverage. The principal of debt matures exponentially at a constant amortization rate $\zeta > 0$, corresponding to an average bond maturity of $\frac{1}{\zeta}$. Therefore, at each instant, firm i needs to repay ζB_t^idt units of maturing bonds at face value. Thus, combining the interest and principal, firm i needs to repay $(\zeta + c)B_t^i$dt in total to avoid default.

Firms can actively manage their leverage by issuing new debt or repurchasing outstanding debt. To adjust their debt level by $D_t^i dt$, they receive proceeds $P_t^i D_t^i dt$, where P_t^i is the bond price specific to firm i at time t, taken as given by firms. For tractability, I assume the newly issued bonds are identical to the existing bonds both in terms of seniority and maturity. I also focus on the recursive equilibrium, so the price P_t^i is only a function of the firm i's state variable (and time), i.e., $P_t^i \equiv P_t(K_t^i, B_t^i)$. To adjust their outstanding debt, firms also need to pay a quadratic debt adjustment cost $\varphi^B(\frac{D_t^i}{K_t^i})K_t^i dt \equiv \frac{\varphi_b}{2}\left(\frac{D_t^i}{K_t^i} - \zeta\right)^2 K_t^i dt$.[9] This adjustment cost captures the debt issuance cost such as underwriting expenses. The law of motion for the outstanding debt is then given as:

$$dB_t^i = \left(D_t^i - \zeta B_t^i\right) dt \tag{3.6}$$

To summarize, during $[t, t+dt]$, firm i produces $Y_t^i dt$, repays $(c+\zeta)B_t^i dt$ to bondholders, invests $\iota_t^i K_t^i dt$, issues bonds $D_t^i dt$ to receive $P_t^i D_t^i dt$ in cash, and pays corporate taxes $\tau\left(Z_t K_t^i - cB_t^i - \delta K_t^i\right) dt$, capital adjustment costs $\Phi_t^{K,i} dt$ and debt adjustment costs $\Phi_t^{B,i} dt$. The remaining is paid out as dividends to equity holders. Hence the dividend flow Π_t^i is given as:

$$\Pi_t^i = (1-\tau)\,\tilde{Z}_t K_t^i + \underbrace{\tau(\delta K_t^i + cB_t^i)}_{\text{tax shield}} - \underbrace{(I_t^i + \Phi_t^{K,i})}_{\text{inv.\&adj}} - \underbrace{(\zeta + c)B_t^i}_{\text{debt repay}} + \underbrace{P_t^i D_t^i - \Phi_t^{B,i}}_{\text{new issu.\&adj}}. \tag{3.7}$$

Default, exit, and entry. The focus of this paper is on the effects of frictions on the credit market. Nevertheless, with a frictionless equity market, equity financing will undo distortions caused by credit market disruption by dipping into the equity holders' pocket. Therefore, I make a stylized assumption on the equity financing, such that firms cannot issue new equity after they are born, which gives a non-negative dividend constraint:

$$\Pi_t^i \geq 0. \tag{3.8}$$

The assumption on equity financing is stylized but realistic as empirically external equity issuance is both costly and infrequent (Gustafson and Iliev, 2017), which is particularly the case during financial crises (Kahle and Stulz, 2013). Empirically the literature has also shown that the credit market is more closely connected to firms' investment decisions than the equity market

[9] The technology of the debt adjustment cost rules out jumps in debt issuance, as jumps will requires an infinite cost.

(Fazzari et al., 1988; Philippon, 2009; Gilchrist and Zakrajšek, 2012). In principle, this model can also allow for costly equity issuance. In that case, the decision of default is strategically chosen by the firm by weighing the continuation value against the outside option, and the default boundary will be characterized by a smooth-pasting condition such as those in Leland (1994). This is computationally more untractable in a general-equilibrium model. The non-negative dividend constraint is commonly used in such general-equilibrium models with financial frictions for efficient computation (Khan and Thomas, 2013; Ottonello and Winberry, 2020).

The non-negative dividend constraint provides another reason for leveraged firms to continue borrowing for financing investment or simply rolling over outstanding debt. However, when the debt burden is too high for a firm, the non-negative dividend constraint cannot be met with full debt repayment. The firm then has to default and declare bankruptcy. Bankruptcy forces the equity holders to walk away from the firm and receive zero payoffs. The creditors seize and liquidate its capital after paying liquidation costs. As staying alive always has a positive option value, firms will never find it optimal to default voluntarily, and will default only if they cannot stay solvent. When earnings are not enough to cover their debt repayments, firms disinvest or issue more debt to stay afloat, until the adjustment costs increase faster than the additional cash they can raise from doing so. The default region is therefore the region of (K, B) where the maximum amount of cash the firm can extract is still not enough to cover the debt repayment, i.e.,

$$\mathbf{R}_t^{df} = \left\{ (K, B) \in \mathbb{R}^2 \Big| \max_{I,D} \Pi_t \left(I, D | K, B \right) < 0 \right\}.$$

In the data, default is a relatively rare event while firms exit for various other reasons. To match the realistic lifespan for firms and avoid counterfactual over-accumulation of capital, I also assume that firms will be hit by an exogenous exit shock at a Poisson rate ξ, corresponding to an average lifespan of $1/\xi$ conditional on non-defaulting. Firms hit by the exit shock liquidate their capital without liquidation costs, repay the debt at face value,[10] and distribute the remaining to equity holders.

To keep the total measure of firms constant, for each exiting firm there is a new entrant endowed with initial capital K_0 and debt level D_0. Entry is exogenous and the net value of entry is treated as a lump-sum transfer to the representative households, so equity holders for existing firms will not take the entry value of the new firms into account when defaulting.

Optimal Policies. After laying out possible actions for the firms, I now turn to the optimization problem faced by firm managers. Firm managers maximize the discounted value of dividends,

[10]When the total face value of the debt D_t^i is higher than capital stock K_t^i, the firm effectively cannot repay the full amount. Under realistic calibration, only a very small proportion of firms are in this region, therefore I omit the discussion for such cases here.

plus the terminal value in the case of an exogenous exit:

$$V_0^i = \max_{I_t, B_t} \mathbb{E}_0 \left[\int_0^{T_{df}} e^{-\int_0^t (r_s + \xi) ds} \left(\Pi_t^i + \xi \max \left\{ K_t^i - B_t^i, 0 \right\} \right) \mathrm{d}t \right]$$
$$\Pi_t^i \geq 0 \tag{3.9}$$

where the dividend is given in (3.7) and subject to the non-negativity constraint (3.8), the laws of motion for K_t and B_t are given as (3.5) and (3.6), and T_{df} is the stopping time when the firm enters the default region.

Define $V_t(K, B)$ as the equity value of a firm with the state variables (K, B) at time t, also recognize that the bond price P_t^i is also a function defined on the same state space, the optimization problem in (3.9) can be written recursively as a Hamiltonian-Jacobian-Bellman (HJB) equation:

$$(r_t + \zeta) V_t(K, B) = \max_{I, D} \Pi_t(I, D|K, B) + \partial_K V_t(K, B) (I - \delta K) + \partial_B V_t(K, B) (D - \zeta B) + \tag{3.10}$$
$$\frac{1}{2} \partial_{KK}^2 V_t(K, B) \sigma^2 K^2 + \xi \max \{K - B, 0\} + \partial_t V_t(K, B) \tag{3.11}$$
$$0 \leq \Pi_t(I, D|K, B) \equiv (1 - \tau) \tilde{Z}_t K + \tau (\delta K + cB) - I - \varphi^K (\frac{I}{K}) K - (\zeta + c) B + P_t (K, B) D - \varphi^B (\frac{D}{K}) K$$

with the boundary condition given by default:

$$V_t(K, B) = 0 \quad \forall \, (K, B) \in \mathbf{R}_t^{df}$$

Proposition 1 establishes scale independence of firms' problem. It shows that firms' optimal policy functions can be characterized with only a single state variable, the book leverage ratio $b \equiv \frac{B}{K}$:

Proposition 1. *If $P_t(K, B)$ is homogeneous of degree zero in (K, B), then the value function $V_t(K, B)$, optimal investment $I(K, B)$ and debt issuance $D(K, B)$ are homogeneous of degree one in (K, B). The default region can be defined in terms of b. Specifically, we have*

$$\begin{aligned} V_t(K, B) &= v_t(b) K \\ I_t(K, B) &= \iota_t(b) K \\ D_t(K, B) &= d_t(b) K \\ R_t^{df} &= \{ b \in \mathbb{R} | \max_{\iota, d} \Pi_t(\iota, d | 1, b) < 0 \}. \end{aligned}$$

Furthermore, with additional regularity conditions, we can show that the default region is characterized by a default threshold $\bar{b}_t$:

Corollary 1. *If $P_t(K,B)$ is homogeneous of degree zero in (K,B) and weakly decreasing and continuous in leverage b, then there exists a threshold $\bar{b}_t$ such that the default region is $R_t^{df} = \{b \in \mathbb{R} | b > \bar{b}_t\}$.*

Proof. See Appendix A.1.

In Appendix A.1, I also verify the homogeneity condition for $P_t(K,B)$ indeed holds in equilibrium. These results greatly simplify the analysis of firms' problem: instead of studying two-dimensional functions V_t, I_t and D_t, from now on we can focus on the average equity value per capital v_t, the investment rate ι_t and the debt issuance rate d_t as functions of a single state variable b within a bounded interval. The law of motion for b can be expressed in terms of optimal policies using Ito's lemma:

$$db = \underbrace{\left(d_t(b) - (\zeta + \iota_t(b) - \delta)\, b + b\sigma^2\right)}_{\mu_t^b(b)} dt - b\sigma d\mathcal{B}.$$

For notational ease, I define $\mu_t^b(b)$ as a shortcut for the drift of b. □

Distribution and default probability. With the evolution of leverage b as well as capital K, we are able to characterize the evolution of firms' distribution with a Kolmogorov Forward Equation (KFE). Notice that the scale independence property does not hold for the distribution—capital quality shocks introduce a negative correlation between leverage and capital. Thus we need to keep track of both state variables. I define $G_t(K,B)$ as the measure of firms for future reference. Furthermore, we can also compute the default probability for each firm at time t over the future h years, denoted as $Q_t^h(K,B)$, or $Q_t^h(b)$ in terms of leverage b by Proposition 1. I leave the characterization of G_t and Q_t^h to Appendix A.2 and A.3.

3.3 Bond Investors

There are three types of bonds on the corporate bond market: a government bond, non-defaulting corporate bonds, and defaulted corporate bonds. I model defaulted bonds in the model so that near-default bonds will also subject to risk-based constraints, as explained below. All investment into the bond market has to be channeled through the representative bond investor. I model the bond investor as a mutual fund, who has zero net worth and manages investment on behalf of households, subject to regulatory constraints. [11] Bonds are exposed to regulations differentially, which I will discuss in detail below.

[11] Alternatively, the financial intermediary can also be modeled as banks, who leverages up by borrowing from the households, such as those in Gertler and Kiyotaki (2010). The current modeling choice simplifies the model by avoiding another layer of leveraging and focuses on the key frictions of interest, risk-based regulations.

Bond payoff structures. The government bond has a very simple payoff structure: it pays a risk-free return at the rate of r_t^G. A non-defaulting corporate bond issued by a firm indexed by (K, B) at time t is priced at $P_t(K, B)$. In Appendix A.1, I verify that given scale independence in firms' policy functions, $P_t(K, B)$ is also homogeneous of degree zero in (K, B). That is, bonds issued by firms with the same leverage but different sizes are identical to the investors. Therefore, in the following discussion, I directly use the bond pricing function $P_t(b)$ defined over leverage b for clarity.

The return to bonds issued by a non-defaulting firm with leverage b has three components: the coupon flow cdt, the change in value $(1 - P_t(b))$ upon maturity for the ζdt share of outstanding bonds, and changes in the price due to changes in leverage b as well as changes in aggregate conditions across time t. That is,

$$dR_t^c(b) = \frac{cdt + \zeta\,(1 - P_t(b))\,dt + dP_t\,(b)}{P_t(b)}$$

$$= r_t^c(b)dt - \partial_b P_t(b) b \sigma d\mathcal{B}_t \tag{3.12}$$

$$r_t^c(b) \equiv \frac{c + \zeta\,(1 - P_t(b)) + \partial_b P_t(b)\mu_t^b\,(b) + \frac{1}{2}\partial_{bb}^2 P_t(b) b^2 \sigma^2 + \partial_t P_t(b)}{P_t(b)} \tag{3.13}$$

where the second equality follows from applying Ito's lemma to $P_t(b)$. Notice that $d\mathcal{B}_t$ is idiosyncratic (superscript i omitted for the ease of notations), so at the portfolio level they are diversified away, and it is the expected return $r_t^c(b)$ that matters for bond investors. Once $r_t^c(b)$ is known, Equation (3.13) gives a partial differential equation (PDE), and together with boundary conditions it pins down corporate bond prices $P_t(b)$.

When a firm hits the default threshold $\bar{b}_t$, it defaults on its bonds and files bankruptcy. Its capital will be liquidated to pay back debt holders. The liquidation is not instantaneous but follows a Poisson process at the rate of ζ^{df}. Before liquidation, defaulted bonds are still held on the balance sheet of financial intermediaries and hence take space for regulatory constraints. This modeling choice is to ensure that defaulted bonds are always priced lower than near defaulted bonds. Without this mechanism, when constraints bind, near-defaulted bonds may have lower price than defaulted bonds, as the former are further discounted due to regulatory constraints while the latter gives an immediate cash payout $\frac{\kappa}{\bar{b}}$. This modeling device has a realistic motivation: in practice the payout to the debt holders is often delayed due to court proceedings. It also induces procyclical prices of defaulted bonds, consistent with empirical observations (Jankowitsch et al., 2014). The details of the defaulted bonds are laid out in Appendix A.5.

Regulatory constraints. The bond mutual fund allocates its assets W_t^B between the government bond W_t^G, non-defaulting corporate bonds W_t^C and defaulted corporate bonds W_t^{df}. Within corporate bonds, it further allocates its demand across bonds with different leverage

b. Denote $x_t^c(b)$ as the density of the demand distribution for non-defaulting corporate bonds, such that $\int x_t^c(b)db = 1$, and $x_t^{df}(b)$ for defaulted corporate bonds. The return to the total assets is:

$$\begin{aligned} dW_t^B &= d(W_t^G + W_t^C + W_t^{df}) \\ &= r_t^G W_t^G dt + W_t^C \int dR_t^c(b) dX_t^c(b) + W_t^{df} \int dR_t^{df}(b) dX_t^{df}(b) \\ &= \left(r_t^G W_t^G + W_t^C \int x_t^c(b) r_t^c(b) db + W_t^{df} \int x_t^{df}(b) r_t^{df}(b) db \right) dt \\ &= r_t^f W_t^B dt \end{aligned} \tag{3.14}$$

where the third equality follows from that idiosyncratic shocks are fully diversified away in the portfolio. Without further frictions, by no-arbitrage all bonds should offer the same expected return, i.e., $r_t^G = r_t^b(b) = r_t^{df}(b') \ \forall (b, b')$. The fourth equality follows from the household's optimization: by no-arbitrage, the return from the bond fund should also be equalized with the return from equity, r_t^f.

However, the bond fund cannot adjust their portfolio freely, constrained by regulations. There are two types of risk-based constraints, a risk-weight constraint for all corporate bonds and a high-yield constraint for high-risk bonds only. Two constraints capture different aspects of regulations in practice. The risk-weight constraint assigns risk weights to all corporate bonds based on their default probabilities. This approach is close to the internal-ratings-based approach as in the Basel Accords reviewed above. Specifically, the regulatory body specifies a weighting function $rw(Q_t^h)$ that maps default probabilities of corporate bonds to risk weights. rw is monotonically increasing in default probabilities, and defaulted bonds are treated as $Q_t^h = 1$. The regulatory body further stipulates a risk capacity proportional to total assets, limiting the risk loading ψ^{RW} in the portfolio, such that[12]

$$\psi_t^{RW} \equiv \frac{W_t^c \int x_t^c(b) rw\left(Q_t^h(b)\right) db + W_t^{df} rw(1)}{W_t^B} \leq \bar{\psi}^{RW} \tag{3.15}$$

This constraint is global in the sense that once it binds, all corporate bonds are affected.

The high-yield constraint specifies a threshold in the default probability $\bar{Q}^{HY}$. Bonds with default probabilities higher (lower) than $\bar{Q}^{HY}$ are classified as high-yield (investment-grade) bonds. The HY constraint requires that the total share of high-yield bonds in the portfolio

[12] As reviewed in Section (2) , in practice, such constraints are often implemented as capital (equity) requirements on banks and insurance companies. As long as financial institutions cannot freely adjust their equity, the capital requirement constraint will be effectively on portfolio risks as in the model. It is well-documented that equity issuance is infrequent and costly, especially during economic downturns.

cannot exceed a certain threshold, i.e.,

$$\psi_t^{HY} \equiv \frac{W_t^c \int x_t^c(b) \mathbb{I}\left\{Q_t^{t+h} \geq \bar{Q}^{HY}\right\} db + W_t^{df}}{W_t^B} \leq \bar{\psi}^{HY}. \tag{3.16}$$

The high-yield constraint produces the market segmentation between investment-grade and high-yield bonds induced by the discontinuity of regulations in practice.

Finally, the government bond is not subject to any regulatory constraint in this model. Therefore, the government bond provides a "safe haven" during a flight-to-safety episode.

Bond pricing. Given the household's discount rate r_t^f and the risk-free nature of the portfolio return, the fund's capital allocation problem is essentially static: it maximizes its instantaneous return subject to the RW constraint (3.15) and the HY constraint (3.16). Denote η_t^{RW} as the Lagrangian multiplier for the former, and η_t^{HY} for the latter, the first order conditions link $r_t^c(b)$ to r_t^G:

$$r_t^c(b) = r_t^G + \eta_t^{RW} rw\left(Q_t^h(b)\right) + \eta_t^{HY} \mathbb{I}\left\{Q_t^{t+h} \geq \bar{Q}^{HY}\right\} \tag{3.17}$$

Finally, the no-arbitrage condition between the bond fund and equity links bond returns to the risk-free return earned from equity:

$$r_t^f = r_t^G + \varphi_t^{RW}\eta_t^{RW} + \varphi_t^{HY}\eta_t^{HY} \tag{3.18}$$

When either constraint binds ($\eta > 0$), there is a positive spread between the prevailing risk-free return in the economy r_t^f and the return to the government bond r_t^G. This spread can be interpreted as a source of the "convenience yield" of the government bond for its regulatory advantage.

With expected returns pin down by (3.17)-(3.18), I can fully characterize the bond pricing scheme. Plug $r_t^c(b)$ back into the full return to corporate bonds (3.12), we obtain a PDE of $P_t(b)$:

$$\left(r_t^c(b) + \zeta\right) P_t(b) = c + \zeta + \partial_b P_t(b)\mu_t^b(b) + \frac{1}{2}\partial_{bb}^2 P_t(b) b^2\sigma^2 + \partial_t P_t(b) \tag{3.19}$$

By value-matching, the bond at the default boundary has the same price of the defaulted bond, $P_t(\bar{b}_t) = P_t^{df}(\bar{b}_t)$., where $P_t^{df}(\bar{b}_t)$ is given by the ODE for the defaulted bond in Appendix A.5. .[13]

[13] Another boundary condition along the dimension b is naturally implied in the formulation of the PDE: at $b = 0$, the second-order term disappear, which gives a standard Robin boundary condition: $\left(r_t^c(0) + \zeta\right) P_t(0) = c + \zeta + \partial_b P_t(0)\mu_t^b(0) + \partial_t P_t(0)$.

3.4 Government

The government is relatively simple during the steady state. It faces the budget constraint such that

$$r_t^G B_t^G + Transfer_t = \frac{dB_t^G}{dt} + Tax_t.$$

In words, it pays out interests to the government bond and lump-sum transfer to households, financed by either further borrowing or revenues from corporate taxes. For simplicity, I further assume that the government commits to a constant debt policy at usual times, so $B_t^G = \bar{B}$.

3.5 Equilibrium

An equilibrium in this economy is defined as paths of aggregate prices $\left\{w_t, r_t^f, \eta_t^{HY}, \eta_t^{rw}, r_t^G\right\}$ household decisions $\{C_t, L_t\}$, firms' policy (functions) $\left\{\iota_t(b), d_t(b), L_t^D(K), \bar{d}_t\right\}$, firms' default probability and bond pricing functions $\left\{Q_t^{t+h}(b), P_t(b)\right\}$, measures of firms $G_t(K, B)$, and aggregate quantities, such that, at every time t:

1. Given aggregate prices, households optimally choose their consumption C_t and labor supply L_t;
2. given bond pricing function $P_t(b)$ and aggregate prices, firms choose policy functions optimally;
3. given firms' policy functions and aggregate prices, the bond fund prices corporate bonds according to (A.4)-(3.19), subject to risk constraints;
4. the evolution of firms' distribution and default probabilities are consistent with firms policy functions;
5. the government budget constraint holds;
6. all markets clear.

There are six markets in our economy: the labor market, the government bond market, the corporate bond market, the defaulted bond market, the equity market, and the goods market. With slight abuse of notations, I use $G(b, K)$ as the measure of firms over the space (b, K) when it is more convenient, $G^x(x)$ as the marginal distribution over the space x, where $x \in \{b, K\}$. I also use g as the corresponding density function of its uppercase counterpart.

The labor market clears when the labor supply from the household equals the total labor demand from firms,

$$L_t = \int L_t^D(K)\mathrm{d}G^K(K). \tag{3.20}$$

The government bond clearing condition is simply the demand by the bond fund equals the supply by the government,

$$W_t^G = B_t^G.$$

The corporate bond market clears when

$$W_t^C x_t(b) = \int b \cdot K \cdot P_t(b) \cdot g_t(d, K)dK,$$

so that for each bond indexed by b, the total amount held by the bond mutual fund equals the market value of bond issued by firms with leverage b. I leave the market clearing condition for defaulted bonds to Appendix A.5 as it requires characterization of the default flow first in A.2.

The equity market clears, such that the total wealth in equity equals the total equity value of firms

$$W_t^E = \int V_t(K, B)KdG_t(K, B).$$

Finally, the goods market clears residually due to Walras's law.

4 Parameterization

Without aggregate shocks, the model yields a steady state. I interpret the steady state as the usual time and calibrate it to match several important moments in the U.S. economy between 2010-2019. Parameters are calibrated with different approaches: Some parameters, for example the depreciation rate, have clear observable empirical counterparts, so they are set directly using empirical moments; some other parameters are taken from the common estimates in the literature; the rest are calibrated internally by minimizing the distances of moments in the model and in data.

The empirical moments are mostly obtained from two sources: the balance-sheet moments are computed from public firms in Compustat using samples between 2010Q1-2019Q4, and moments regarding the credit market are taken from Moody's default and recovery database. To match model moments to data, I interpret K^i as total assets of a firm and B^i as total debt (short term debt + long term debt). To capture the realistic quantitative significance, I target on size-weighted moments in data whenever possible.

Parameters in this model fall into two blocks: *the financial block*, including parameters governing the credit market and regulations, and *the macroeconomic block*, including parameters governing consumption, production, and firm dynamics. Below I discuss the calibration for these two blocks respectively.

The financial block. It is of crucial importance for the model to capture the realistic level of default risks and its distribution across firms. In data, default is a relatively rare event: the average one-year default probability in data according to Moody's default report is around 0.5%, dollar-weighted. Conceptually, default probabilities are affected by two factors: how far an average firm is away from default, and how fast the firm diffuses. For the former, I match the average (size-weighted) book leverage ratio to data (0.32 in Compustat) by calibrating the debt issuance cost parameter to $\varphi_b = 4.13$, and for the latter I calibrate the capital quality shock volatility $\sigma = 0.2$ to hit the dollar-weighted default rate. An important stylized fact on the credit market is that the high-yield market is relatively small in the U.S. According to the Mergent FISD bond issuance data, in 2019 high-yield bonds account for only 15% of the total outstanding corporate bonds in face value. I calibrate the IG/HY cutoff in one-year default probability $\bar{Q}^{HY}$ so that in the model-generated stationary distribution, 15% of outstanding bonds are classified as high-yield and subject to the HY constraint. As a result, the dollar-weighted default probability for HY bonds is 3.33% (p.a.) in the model. It is close to its empirical counterpart, which ranges between 2.0%-4.0% during normal times. At the steady state, the cutoff for HY bonds in terms of leverage ratio is at $\bar{b}^{HY}_{SS} = 0.52$. Notice that $\bar{b}^{HY}$ is not constant across transition dynamics as the default probability changes with aggregate conditions even for firms with the same leverage ratio.

I assume the risk-weighting function $rw(Q)$ is a continuously increasing function of one-year-forward default probability, which takes the form of

$$rw(Q) = a_{rw} + b_{rw} Q^{\psi_{rw}}, \tag{4.1}$$

where $a_{rw}, b_{rw}, \varphi_{rw} > 0$ are parameters to be determined. I calibrate the rw using the risk weights in the standardized approach in Basel II for banks, while the RBC for life insurance companies essentially give very similar weights. [14] The Basel Accord gives a 20% risk weight to corporate claims with the highest rating (AA- or above), a 100% risk weight to claims of marginal HY firms (BBB to BB-), and a 150% for claims with the lowest rating (below BB-). These provide three data points for the rw function, $rw(0) = 0.2$, $rw(\bar{Q}^{HY}) = 1.0$, and $rw(1) = 1.5$, so coefficients

[14] After the update in 2019, the weighting scheme for life risk-based capital (RBC) is essentially very close to that in the Basel Accord. If the weight on the marginal HY bonds is normalized to 100%, their weighting scheme gives the same weight as the Basel Accord for the riskiest bonds (150% for defaulted bonds), and 15% for the safest bonds, slightly lower than that in Basel III.

for rw are fully pinned down. Notice the unit of rw does not matter as long as it scales together with the risk capacity constraint $\bar{\psi}^{RW}$.

Figure 1 plots the credit market across leverage ratio b. Panel (a) shows the distribution of bonds across leverage ratio b. The light area (left to the cutoff) displays the share of IG bonds, and dark area (right) displays the share of HY bonds, which integrates to 15% by calibration. At the steady state, firms default $\bar{b}_{ss} = 1.48$. As the default boundary $\bar{d}$ is absorbing, no density can be accumulated at the boundary, and density diminishes as b moves towards the default boundary. Panel (b) presents risk weights at the steady state as a function of leverage. Risk weights are computed from function (4.1). The input of the risk-weighting function, one-year forward default probability, is plotted on the right y-axis. Even though IG bonds have close-to-zero one-year default probability as in practice, the calibrated risk-weighting function is very sensitive to small default risks and therefore have a steep curve to the left. For reference, I also plot the five-year-forward default probability on the right y-axis. The marginal HY bonds have a non-trivial default probability in five-years, at around 5%.

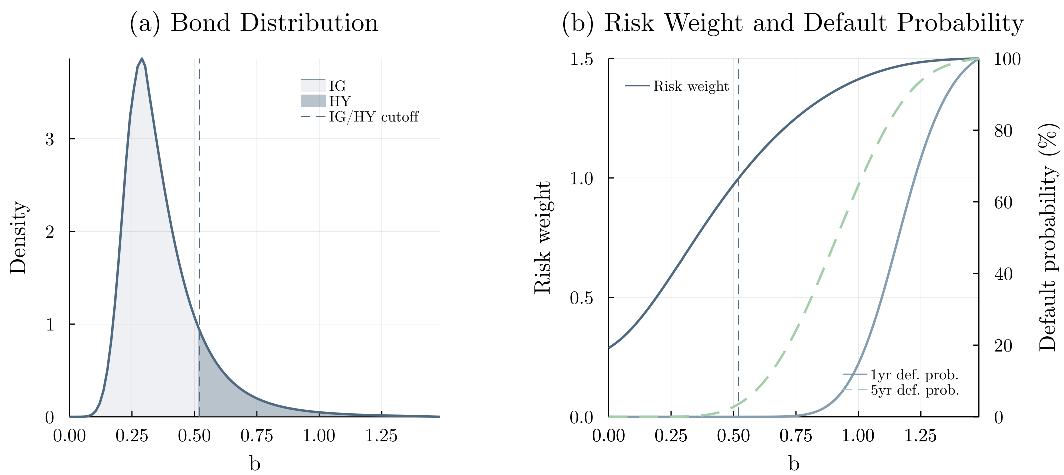

Figure 1: Credit Market

The risk-weight constraint $\bar{\psi}^{RW}$ is set to be marginally binding at the steady state. In other words, $\bar{\psi}^{RW}$ is calibrated to the risk loading of the bond fund at the steady state $\psi^{RW}_{SS} \equiv \frac{W^c_{SS}\int x^c_{SS}(b)rw\left(Q^h_{SS}(b)\right)db+W^{df}_{SS}rw(1)}{W^B_{SS}}$. Therefore, the Lagrangian multiplier during the steady state $\eta^{RW}_{SS} = 0$, but upon any negative shock the RW constraint will bind, and the intended mechanism will be activated. This choice reflects two considerations. On one hand, the steady state captures the long-run equilibrium, or the "usual time" when the financial market functions frictionlessly; on the other hand, we have considerable empirical evidence that even during usual times there exist non-credit-risk components in corporate bond yields[15], part of which may reflect the discounts due to risk-based regulations. Consistent with what this model would predict, the non-credit-risk components in credit spreads also increase with credit risks (He and Milbradt, 2014; Chen et al., 2018). I pick the middle ground here by setting the risk-weight constraint to be marginally binding. The high-yield constraint $\bar{\psi}^{HY}$ is generally not binding at the usual time for insurance companies and mutual funds (Ellul et al., 2011; Chernenko and Sunderam, 2012). I experiment with different levels of the HY constraint for transition dynamics to understand how the high-yield constraint affects the financing and investment of firms with heterogeneous credit risks.

Other parameters in the financial block are calibrated as follows. The initial leverage for a newborn firm is set to 0.29 according to Compustat. The calibration of the debt security generally follows the literature (Demarzo and He, 2020; Gomes and Schmid, 2020). I calibrate the debt amortization rate ζ to be 0.2, corresponding to an average maturity of 5 years. The coupon rate is the same as the steady-state interest rate $c = 4.77\%$, so the price of a risk-free bond price is equal to 1 at the steady state. The average bankruptcy resolution duration is set as one year, i.e., $1/\zeta^{df} = 1$. Upon resolution, capital is recovered at the rate of $\kappa = 0.7$. Both parameters are calibrated from Moody's default and recovery database.

Table 1 summarizes the calibrated parameters for the financial block.

Table 1: Parameterization: the Financial block

Parameters	Description	Value	Target/source
σ	Capital quality shock vol.	0.2	1yr default rate 0.5%
φ_b	Debt issuance cost	4.13	Average book leverage ratio $\bar{b} = 0.32$
d_0	initial leverage	0.29	leverage ratio for newly listed firms
ζ	Bond maturing rate	0.2	Average maturity 5 years
c	coupon rate	4.77%	Risk-free bond price $P = 1$
$\bar{Q}^{HY}$	HY threshold in 1yr default prob.	1.5×10^{-6}	High-yield share in face value $\frac{B^{HY}}{B^{IG}+B^{HY}} = 15\%$
κ	Recovery rate of capital	0.7	Moody's default report December 2019
$1/\zeta^{df}$	Avg. recovery period for defaulted bonds	1 (yr)	Moody's default and recovery database
$\bar{\psi}^{RW}$	Risk-weighted constraint	0.371	marginally binding at S.S. $(\eta^{RW} = 0)$
$\bar{\psi}^{HY}$	HY constraint	0.085	Impulse response

Notes. Rates are expressed in annualized values.

The macroeconomic block. I keep the calibration of the macroeconomic block as close to the literature as possible. Thanks to scale independence, the aggregate economy preserves the form of Cobb-Douglas function: $Y_t = Z_t L_t^{\alpha} K_t^{1-\alpha}$. I normalize annual output $Y_{ss} = 1$ as the numéraire. I set $\alpha = 0.6$ so the labor share in the economy is 60%. I further also set the steady state labor supply to be 1 by calibrating the labor disutility parameter $\beta = 1.16$. This further implies the wage at the steady state to be $w_{ss} = \frac{\alpha Y_{ss}}{L_{ss}} = 0.4$. TFP at the steady state Z_{ss} is calibrated to

[15] There is a large literature on the liquidity component of credit spreads. See for example Longstaff et al. (2005); Ellul et al. (2011); Mota (2021); He and Milbradt (2014); Chen et al. (2018); Li and Yu (2021).

0.64, so firms' earnings per unit of capital $\tilde{Z}_{ss} = 0.133$, consistent with the empirical estimate from Compustat. With Y_{ss},Z_{ss}, and L_{ss} determined, the aggregate capital stock is also pinned down at $K_{ss} = 3$. I set the initial capital for entrants $K_0 = 0.419$ so the aggregate capital stock is consistent with that implied from the stationary distribution.

The household discount rate is calibrated to $\rho = 4.77\%$. The more patient households are, the higher investment firms are willing to make. At the steady state, the average growth rate of firms is 5.5% as its empirical counterpart in Compustat. The depreciation rate $\delta = 4.5\%$ is directly calibrated from Compustat as the ratio of depreciation and amortization to total assets. Finally, the exogenous exit rate for firms is set to $\xi = 0.0667$, corresponding to an average lifespan of 15 years for public firms. Several other parameters are common in the literature and therefore calibrated accordingly.[16] Table 2 summarizes the calibrated parameters for the macroeconomic block.

Table 2: Parameterization: the Macroeconomic Block

Parameters	Description	Value	Target
Preferences			
$1/\gamma$	IES	1.0	Literature
$1/\nu$	Frisch elasticity	1.0	Literature
β	Labor disutility	1.16	Normalize labor supply to 1at S.S.
ρ	Discount rate	4.77%	Average asset growth rate of 5.5%
Technology			
α	Labor share	0.6	Literature
Z_{ss}	TFP at the S.S.	0.64	EBITDA/asset ratio 0.133
ϕ_k	Capital adj. cost	4.0	Literature
δ	Depreciation rate	0.045	Depreciation to asset in Compustats
Entry and Exit			
ξ	Exogenous exit rate	0.0667	Average lifespan 15 years
K_0	Capital for entrants	0.419	Capital stock at the S.S. $K_{ss} = 3$
Government			
$\bar{B}^G$	Gov. debt at the S.S.	1.0	debt/GDP around 100% in 2019
τ	Corporate tax rate	0.25	Literature

Notes. Rates are expressed in annualized values. If not otherwise specified, firm-level moments are sourced from public firms in Compustat from 2010Q1-2019Q4.

[16]These are the intertemporal elasticity of substitution $\frac{1}{\gamma} = 1$, the Frisch elasticity of labor supply $\frac{1}{\nu} = 1$, capital adjustment cost parameter $\varphi_k = 4$, corporate tax rate $\tau = 0.25$. Similar calibrations can be found in, e.g., Philippon (2009); Gomes et al. (2016); Kaplan et al. (2020); Gomes and Schmid (2020).

5 Numerical Experiments

I organize my numerical experiments around different policy and regulation scenarios. In Section 5.1, I hit the model with a crisis shock and study the transition dynamics. The crisis shock is composed of a productivity shock and an uncertainty shock, mimicking the COVID-19 recession in March 2020.[17] I consider the transition dynamics under different scenarios where the constraints are activated one by one. In Section 5.2, I further consider a scenario where the government also announces at $t = 0$ an unexpected plan of interventions in the credit market. I use this scenario to evaluate the effect of the unprecedented corporate bond purchase programs conducted by the Federal Reserve in March 2020. I also consider alternative policies in comparison.

5.1 Transition Dynamics under Different Regulation Scenarios

Now I shock the model at the steady state with an unanticipated probability-zero crisis. The crisis is modeled as a sudden drop in aggregate TFP Z_0 together with an increase in the variance of idiosyncratic shocks σ_0^2. After the initial impact, the evolutions of Z_t and σ_t^2 are deterministic and perfectly anticipated. In other words, it is an "MIT" shock. I calibrate the shocks so that they mimic the market's expectation at the beginning of the COVID-19 recession in March 2020. Specifically, the TFP Z_0 drops by around 10% to match the 10% drop in GDP in 2020Q2, and the volatility σ_0^2 jumps up by 20% to match the average high-yield default probability of 6% at the peak of the crisis. Both shocks persist at the initial level for one year, and then mean reverts following an Ornstein-Uhlenbeck process:

$$
\begin{aligned}
dZ_t &= \rho \left(Z_{ss} - Z_t\right) dt \\
d\sigma_t^2 &= \rho \left(\sigma_{ss}^2 - \sigma_t^2\right) dt,
\end{aligned}
$$

where ρ is set to 1.5 so that the shocks almost completely die down after 4 years. Figure 2 plots the shock paths during transition dynamics, expressed as deviations from steady state values.

I consider three regulatory scenarios with constraints activated one by one: an unconstrained scenario, where both constraints are turned off, an RW-constrained scenario, where the risk-weight constraint is in effect while the high-yield constraint is slack, and finally an RW&HY-constrained scenario where both constraints are active. The unconstrained scenario can be

[17]Due to the multifaceted nature of the COVID-19 shock, the productivity and uncertainty shock in this model is only a reduced-form simplification. A full characterization of the economy during the pandemic falls beyond the scope of this paper. See e.g. Guerrieri et al. (2020); Kaplan et al. (2020); Alvarez et al. (2021) for studies on other aspects of the pandemic-induced recession.

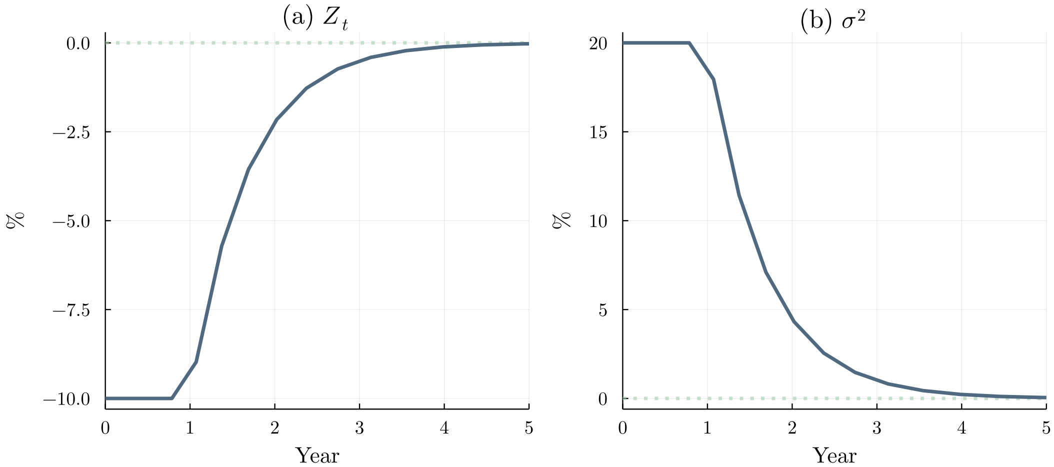

Figure 2: Shocks during transition dynamics

interpreted as the counterfactual of a regulatory regime without risk-based constraints, or the effect of a countercyclical regulatory regime, which automatically relaxes the constraints in economic downturns. [18]

Figure 3 shows the . the risk loading (ψ^{RW}) in panel (a) and the share of high-yield bonds in the portfolio (ψ^{HY}) in panel (b) during the transition under different scenarios. Under the unconstrained scenario (dotted green lines), both the risk loading and the HY share shoot up upon the crisis shock. Default probabilities endogenously increase due to a lower productivity and a higher uncertainty, so bonds have higher risk weights conditional on the same states (leverage ratio b), and some bonds previously near the high-yield threshold are downgraded immediately after the shock. Under the RW-constrained scenario (dashed blue lines), the risk-weight constraint $\bar{\psi}^{RW}$ is set at the steady state level, limiting the total risk loading of bond investors. To fit in the limited risk capacity, bonds with higher risk weights are traded at larger discounts, as will be shown below. The risk-capacity constraint also reduces the share of high-yield bonds in the portfolio compared to the unconstrained case, since high-yield bonds are assigned with larger risk weights and hence larger discounts. In the RW&HY-constrained scenario (solid blue lines), the HY constraint is set to $\bar{\psi}^{HY} = 0.85$ so that it binds for the first quarter after the shock.

Figure 4 displays the impulse responses in the credit market. Under the unconstrained scenario, the responses are relatively mild: the default threshold drops very minimally upon impact.[19]

[18] For example, Basel III include a countercyclical capital buffer which may put in place a larger capital requirement for banks when national authorities determine that the credit growth is excessive. It has not been activated in most jurisdictions including the U.S.

[19] When the default threshold unexpectedly drop, firms with leverages higher than the new default threshold

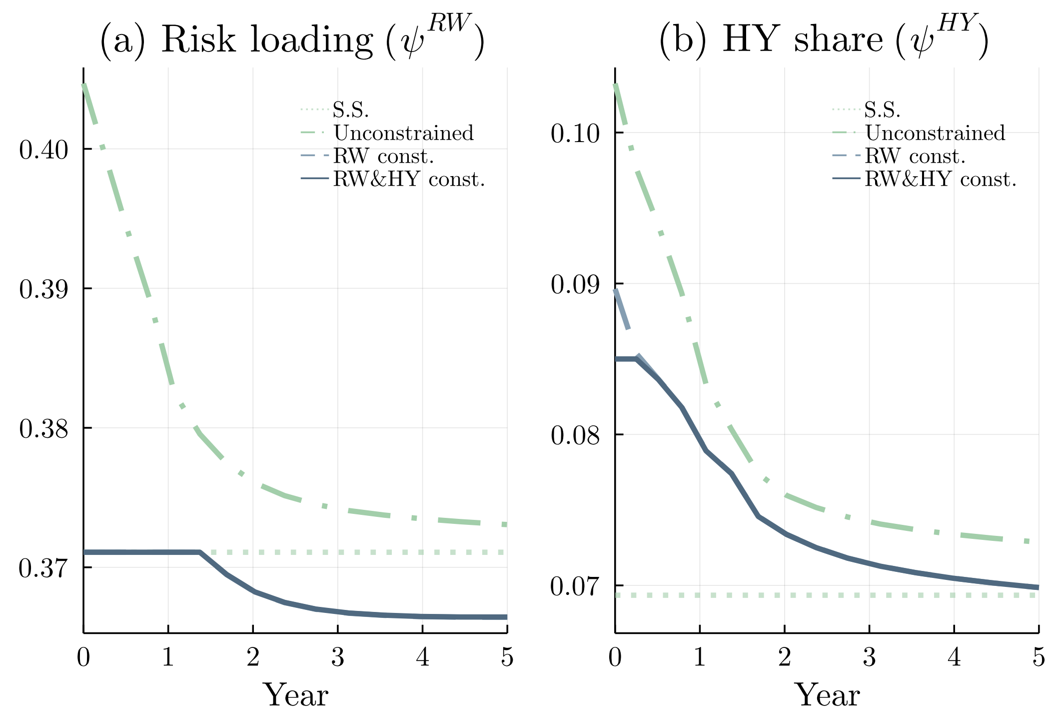

Figure 3: Shocks during transition dynamics

The high-yield threshold in terms of the leverage ratio is more responsive, falling from 0.52 to 0.46. The average default probability increases from the steady state value of 0.5% to 0.65%, and the average default probability for the high-yield bond increases to 4.4%. The responses from yields are positive but also small.

The constraints have profound impacts on the credit market. With the risk-weight constraint in place (dash light-blue lines), the average yields jump to around 8% upon shock for the investment grade bonds (panel c), and 15% for the high-yield (panel f). Higher yields further exacerbates the financial condition of firms, resulting in higher default probabilities. Defaulted bonds are subject to the strictest constraint and take the largest haircuts, so for near-default firms it is harder for them to roll over their debt. The default threshold is therefore significantly lower.

The impact of the RW constraint is highly heterogeneous across firms. The increases in yields and default probabilities are much smaller for IG firms relative to those for HY firms. The additional HY constraint further increases inequality across firms. With both constraints binding (solid dark-blue lines), high-yield bonds are further discounted to meet both constraints. The average yield for high-yield firms increases from 15% to 17.5%. However, the average yield for IG bonds is actually *reduced* relative to the single-constraint case. The reduction in the yield of IG

will default immediately. I assume a same measure of firms are also reborn immediately with the same initial capital and leverage to keep the total measure constant. As shown in Figure 1, the share of firms close to the default boundary is very small so different treatments of the immediate default is not quantitatively important, even under the constrained scenarios.

bonds is the force of a simple general-equilibrium mechanism: when the demand for a good is suppressed, the price for its substitute increases. In this model, this force manifests itself through the no-arbitrage condition, which requires the total bond portfolio to offer an average return of r_t^f. Thus, when HY bonds offer a higher return, IG bonds have to offer a lower return to balance it out.

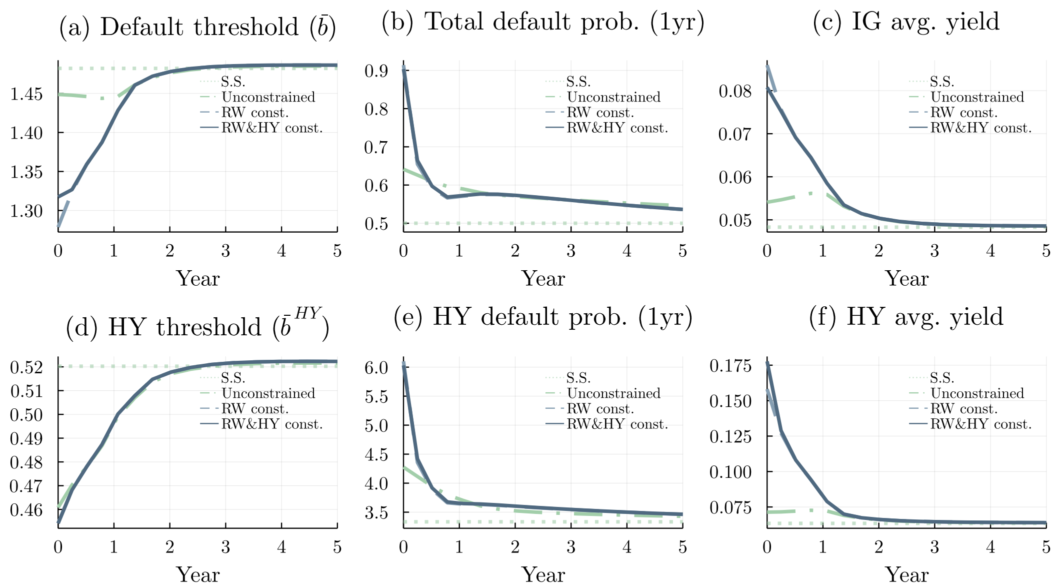

Figure 4: The Credit Market under Different Regulatory Scenarios

To further shed light on the vast heterogeneity across firms, I plot firms' value and policy functions upon impact in Figure 5.[20] Before commenting on the responses upon impact, it is helpful to discuss the overall patterns at the steady state (dotted green lines) as the baseline. The equity value per unit of capital (Panel a) decreases as leverage increases, since highly leveraged firms face higher probabilities of defaulting and exiting with zero values. In parallel, the bond yield increase with leverage, reflecting higher credit spreads due to default risks. Panel (c) plots the rate of bond issuance. Firms with higher leverage issue more debt for two reasons: they have more maturing debt to roll over. To finance the same level of investment, they also need to issue more debt since their bonds are cheaper. Panel (d) displays the investment rate. Firms with relatively low leverage ($b < 0.3$ at the S.S.) are financially unconstrained so their investment is close to the first-best level. As leverage increases, firms cut off investment for two reasons: The financial constraint and high yields are prohibitive for investment; close-to-default also makes investment less attractive to the equity holder, because if the firm defaults, the benefits of additional capital are captured by the debt holders. Instead, highly leveraged firms prefer to disinvest and pay out dividends. This is the well-known *debt overhang* mechanism.

[20] I cut the x-axis at 1.0 for visibility. As shown in the distribution, there are very few firms with leverage higher than one, so their responses are not influential for aggregates.

Upon impact, the shock pushes down equity values and investment. Compared to the unconstrained counterpart (green dash-dot lines), in the constrained scenario (blue lines), the equity value function v rotate clockwise, i.e., the equity values for low-leverage firms are increased by constraints while those for high-leverage firms are decreased. Effectively, safe firms are subsidized by constraints while risky firms are suppressed. The yield function gives a more explicit indication of the redistributional effect of constraints. With the RW constraint, though most firms are facing higher yields on the credit market, the yields for the safest firms are actually lower; some of them even face lower yields than the steady state. The HY constraint adds another distortion to the credit market. The yield start at a lower level for IG firms, and as leverage approaches the high-yield threshold, the yield increases steeply and exceeds the single-constraint counterpart. The change in the financing cost is reflected in firms' policy functions in the lower panels. With binding constraints and therefore a higher financing cost, most firms cut off borrowing and investment. However, the safest firms even increase their investment back to the steady-state level.

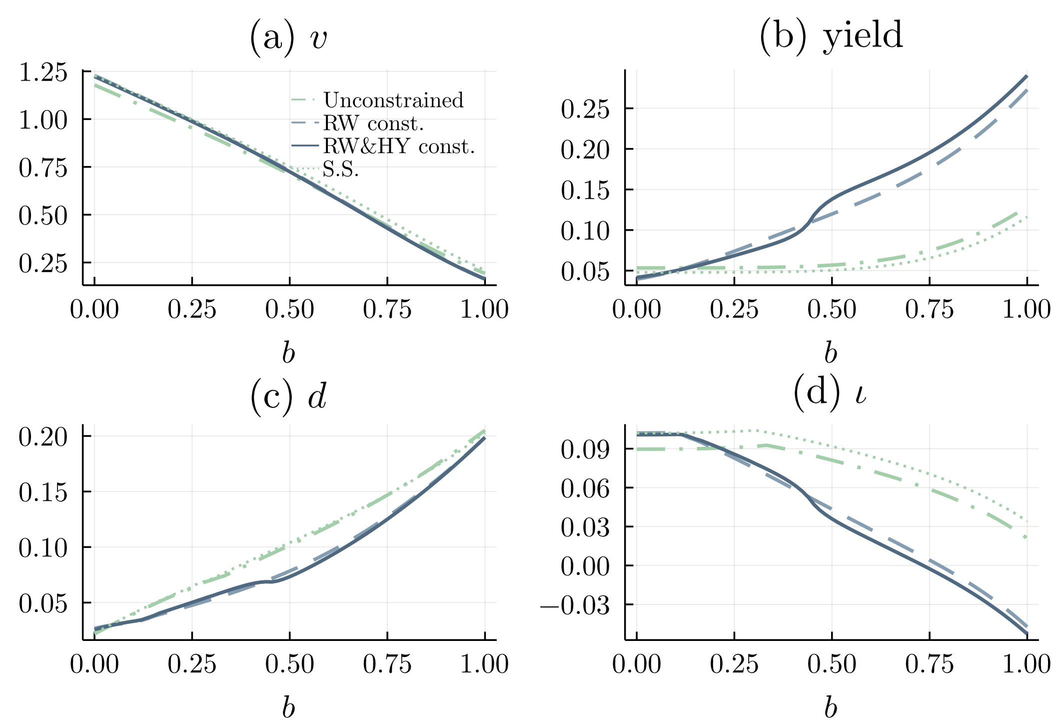

Figure 5: Value and Policy Functions upon Impact

I proceed to evaluate the aggregate effect on investment. Figure 6 displays the time paths of investment by ratings, as percent deviations from the steady state. I define AAA as the safest firms in the whole distribution, i.e., firms with leverage $b = 0$. Under the unconstrained scenario, investment drop similarly by 12-15% for all three ratings. Consistent with the policy function plotted above, the RW constraint increases investment of AAA firms at the expenses of other IG and HY firms. IG firms invest 13% less than the unconstrained scenario, and HY firms invest

60% less. As the shock dies down, constraints become less binding, and paths of investment gradually converge back to the unconstrained scenario in one year. The HY constraint further suppresses HY firms' investment by 10%, and slightly improves that by IG firms.

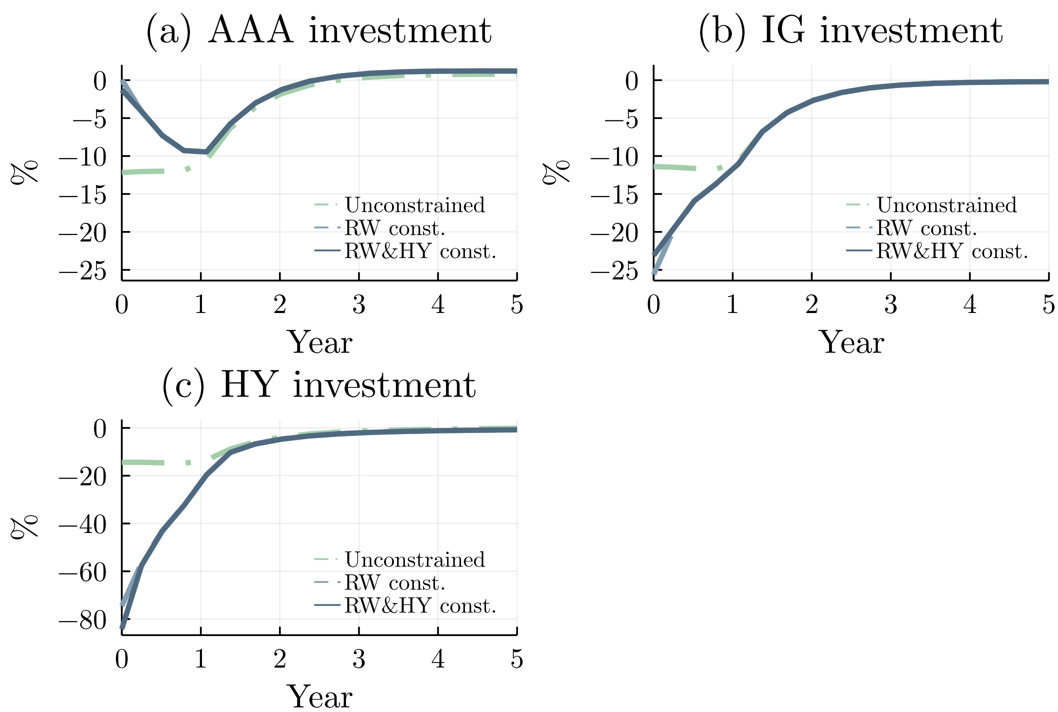

Figure 6: Investment by Ratings

Figure 7 displays the percent deviations of aggregate variables. The risk-weight constraint has a quantitatively large effect on aggregate investment: the investment drops by 12% under the unconstrained scenario, and more than 25% under the risk-weight constraint. The high-yield constraint, on the other hand, has a very minimal incremental effect. If anything, it *increases* aggregate investment. As I show above, the HY constraint redistributes from high-yield firms to investment-grade firms. High-yield firms have lower investment propensities compared to investment-grade firms due to debt overhang. Therefore, by financially suppressing the low-investment-motive firms and subsidizing the high-investment-motive firms, the HY constraint increases the total investment. The effect is less noticeable, since by calibration high-yield firms have a small market share.

Panel (b) displays the cumulative effect of the reduction on investment. Constraints lower the capital stock in one year by 0.5% compared to the unconstrained case. The disinvested capital is eventually consumed due to market clearing, leading to a higher consumption in the constrained cases. A higher consumption reduces labor supply and therefore total output, which drops by 10% under the constrained scenario, compared to 7.5% without constraints.

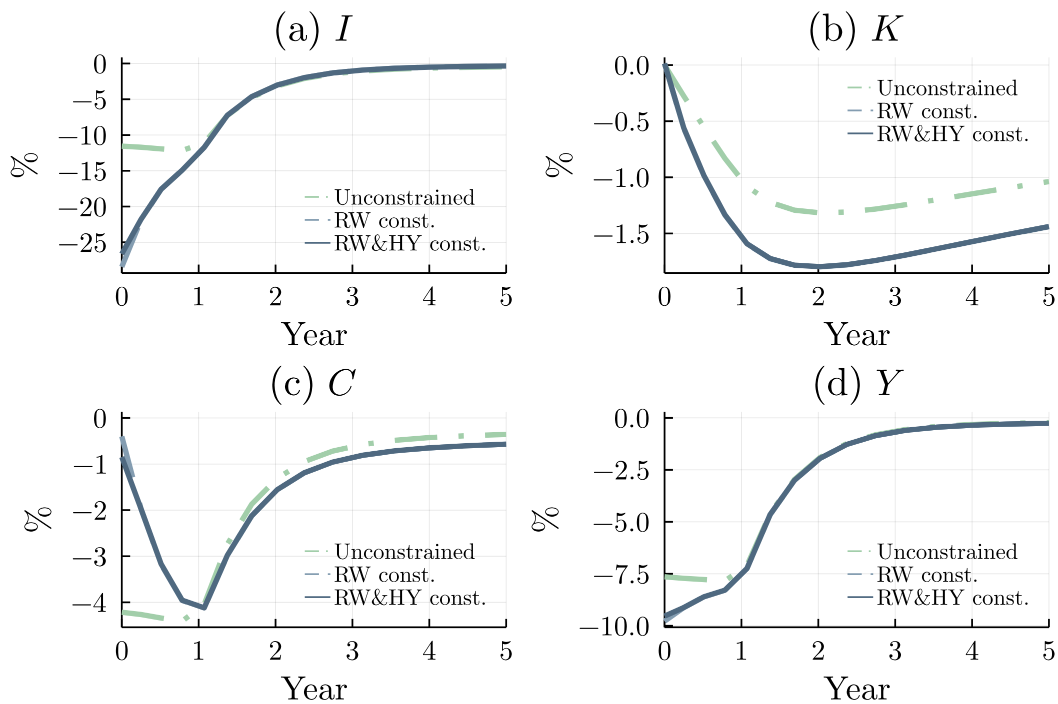

Figure 7: Transition Dynamics of Aggregates

5.2 Credit Market Intervention

In this section I evaluate the effect of the credit market intervention programs announced by the Federal Reserve in March 2020. In response to severe credit market meltdown, the Federal Reserve created two credit facilities to support the credit market: the Primary Market Corporate Credit Facility (PMCCF), which directly made loans to companies, and the Secondary Market Corporate Credit Facility (SMCCF), which purchases corporate bonds or corporate bond ETFs from the secondary market. Both CCFs target on investment-grade bonds and the combined authorized size is up to 750 billion.

Through the lens of this model, the credit market intervention works through relaxing the risk constraints of financial intermediaries. As this model does not feature the distinction between the primary and the secondary markets, I bundle these two facilities together in the following analysis. The intervention programs are modeled as the government purchasing corporate bonds financed by the government bond. The government made an unanticipated announcement on the intervention plan at $t = 0$ when the shock hits. The intervention program can be described by two objects: the Fed's demand function for corporate bonds $x_t^{fed}(b)$, and the outstanding amount of bond purchased by the Fed B_t^{fed}. The government budget constraint is modified as:

$$r_t^G B_t^G + \frac{dB_t^{fed}}{dt} + Transfer_t = \frac{dB_t^G}{dt} + Tax_t + B_t^{fed} \int x_t^{fed}(b) r_t^c(b) db.$$

The purchase schedule is as follows. Starting from $t = 0$, the government linearly increases B_t^{fed} from 0 until reaching the maximum capacity $\bar{B}^{fed}$ at $t = 0.5$, continues to maintain at the maximum capacity until $t = 2$, and then slowly winds down the portfolio at a constant rate until $t = 5$. The maximum capacity is 750 billion, or $\bar{B}^{fed} = 0.375$, with the annual GDP as the numéraire. To mimic the demand function of the Fed in practice, I set $x_t^{fed}(b)$ proportional to the market share of bonds in the investment grade at the steady state. [21] Figure 8 displays the purchase schedule by the Fed. I further assume that, as before, the government commits to a fiscal policy that fixes the level of net liability, i.e., $B_t^G - B_t^{fed} = \bar{B}$. The government earns spreads between constrained corporate bonds and the government bond, and it transfers the proceeds to the household in a lump-sum fashion.

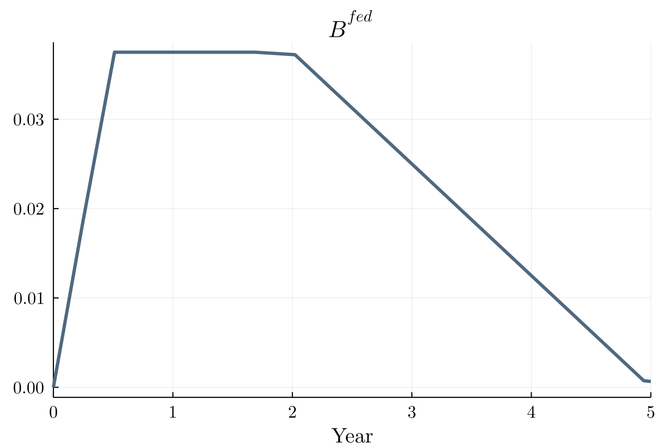

Figure 8: The Path of the Outstanding Amount Purchased

I use the RW-constrained scenario as the laissez-faire scenario, and study the additional effect of credit facilities. Figure 9 shows the responses in the credit market. The stimulus generates a strong anticipation effect. At time 0 under the stimulus scenario (solid blue lines), the government announces the stimulus plan but has not purchased any bond yet. Still, the one-year forward default probabilities are reduced upon the announcement relative to the laissez-faire scenario, as investors rationally expect that the stimulus relaxes the RW constraint, and improves the credit market condition in the near future. Lower default probabilities lead to lower risk weights, which further relaxes the RW constraint, as predicted by the accelerating mechanism. Therefore, the average yields in the credit market drop immediately upon the announcement, relative to laissez-faire counterparts.[22] The effects of stimulus continue to build up in the first

[21] Initially, both CCFs were only open to investment-grade companies. On April 9, the Fed expanded the eligibility to "fallen angels", i.e., firms that were rated as investment grade in March 22rd, 2020. The SMCCF purchases ETFs and bonds based on the broad market index to obtain broad exposure to the market for U.S. corporate bonds.

[22] Such announcement effects are also observed in reality. After the announcement of intervention plans on March 23rd, 2020, the investment-grade credit spreads narrowed 20 basis points on the day, and continue to fall 100 basis points in the next two weeks. The real purchase only started 2 months later. See Gilchrist et al. (2020) for a more careful empirical analysis on the effects of SMCCFs.

year as the Fed increases purchases.

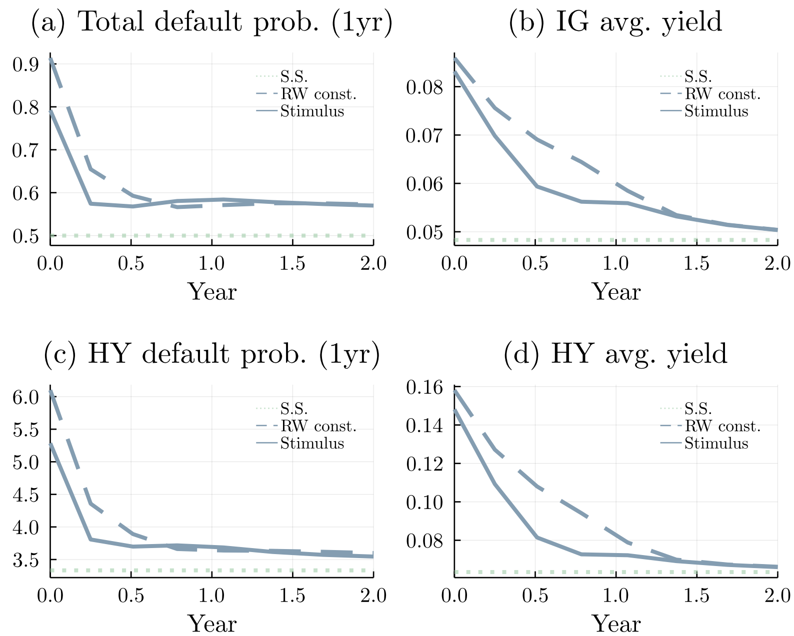

Figure 9: Shocks during transition dynamics

In aggregate, I find that the stimulus significantly speeds up the recovery from the crisis, as shown in Figure 10. The drop in investment upon shock is smaller with stimulus, and it returns to the steady state level more quickly. The faster recovery in investment also speeds up the recovery in total outputs and reduces the dip in capital stock.

Counterfactual policies. I also evaluate whether alternative policies can achieve better outcomes. In particular, I consider a purchase plan that targets only on high-yield bonds. There are several reasons why this policy may have its merits. First, high-yield bonds have higher risk weights, so with the same amount of stimulus, purchasing high-yield bonds can be more efficient at reducing the risk loading of the bond investors. In other words, it may have better "bang for the buck". Second, the high-yield constraint may also bind at the peak of the crisis, which has local effects on high-yield firms and cannot be alleviated by an IG-targeting policy. Hence below I compare two policies with the same path of B_t^{fed} but different demand functions

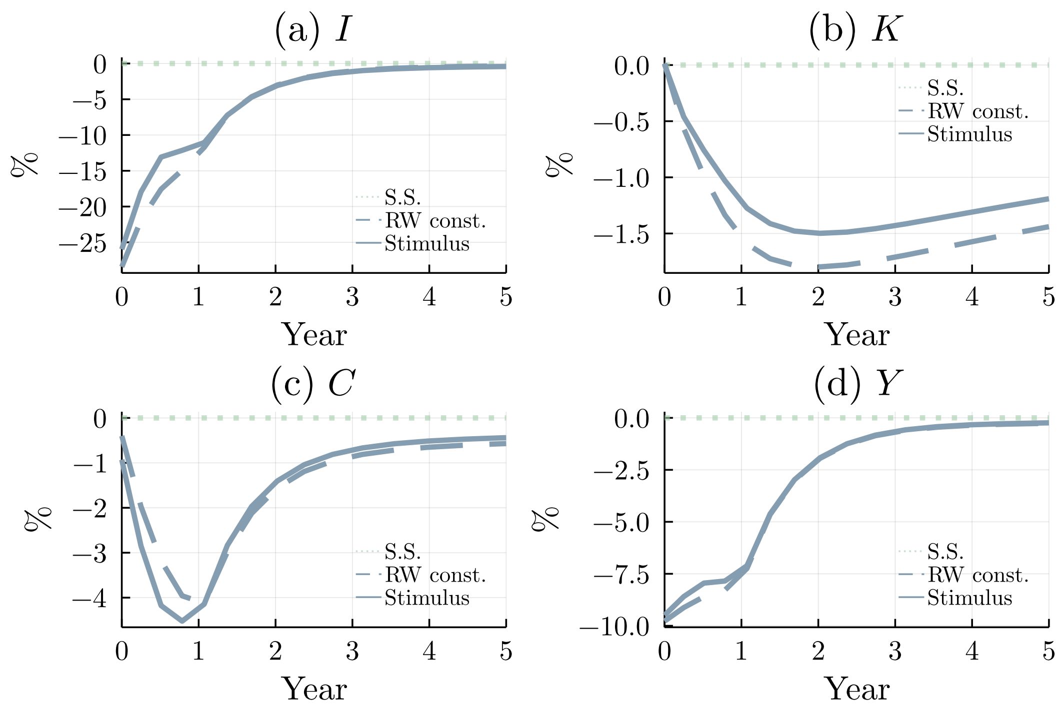

Figure 10: Shocks during transition dynamics

$x^{fed}(b)$. The laissez-faire benchmark here has both constraints binding to allow for the local effect on HY firms.

My quantitative analysis shows that the gain from an HY-targeting policy is very negligible in terms of aggregates. It indeed lowers the yields for high-yield bonds more than does the IG-targeting policy (panel d of Figure 11), but it has hardly any effect on the investment-grade yields. As shown in previous section, it is the IG firms whose investment matters for aggregates, as they are quantitatively large, and sensitive to financial costs. HY firms, on the contrary, are less sensitive to financing costs, and quantitatively have a much smaller role in the aggregate economy. Indeed, as shown in Figure 12, even though investment by high-yield firms drops slightly less under the HY-targeting policy, it does not affect aggregate investment and output in any significant way. In this model, there is no additional cost for purchasing HY bonds versus IG bonds, as all idiosyncratic shocks are diversified away in the portfolio. In real world, however, purchasing high-yield bonds will let the Fed be exposed to more aggregate risks. Given the small gain, a policymaker who cares mostly about aggregates may not find the HY-targeting policy particularly appealing.

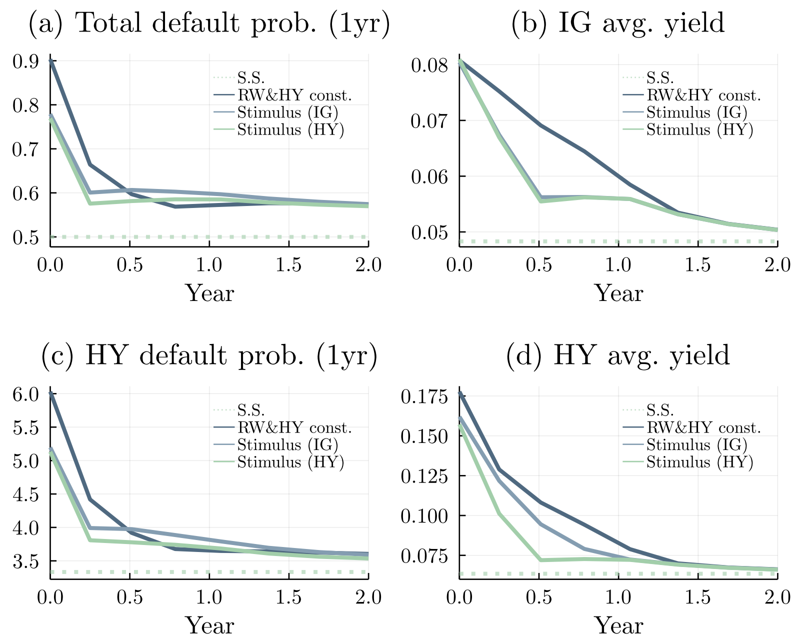

Figure 11: The Credit Market under Different Policy Rules

6 Conclusions

In this paper, I study the role of risk-based regulations in the macroeconomy. I find that risk-based regulations amplify the economic downturn like the classic financial accelerator. The drop in aggregate investment increases from 10% under the unconstrained scenario to 25% under the constrained scenarios. Importantly, effects are heterogeneous across firms: Risky firms whose bonds are more tightly constrained take larger hits from binding constraints than safe firms; the safest firms are even subsidized by such constraints. I use this framework to study the credit market intervention policies as conducted by the Fed in March 2020. I find that the intervention policies speed up the recovery of the economy by relaxing the risk constraints.

It should be cautioned that even though I evaluate the outcomes of regulations and stimulus policies, results here should be interpreted as positive instead of normative. As this model does not feature aggregate risks and all idiosyncratic shocks are diversified away at the aggregate level, there is no endogenous reason for the mutual fund to be regulated. The benefits of risk regulations, such as retail investor protection, micro- and macro-prudential motives, etc., are outside of the model. This model is able to evaluate the cost of such regulations, but does not directly answer the question whether they should be implemented at the first place. Similarly, the costs of government interventions are also outside of the model, so I only evaluate the potential benefits of the intervention policies but not the trade-offs. Nevertheless, I believe a positive analysis of policies can still be informative in guiding future policy designs. I leave a normative analysis for future work.

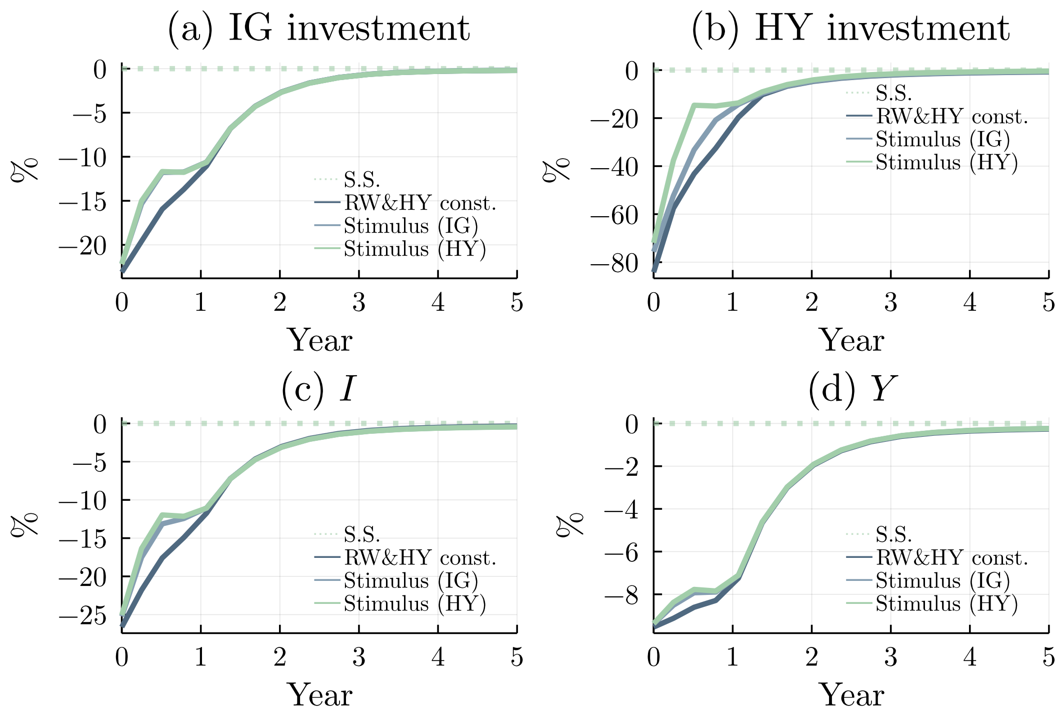

Figure 12: Aggregates under Different Policy Rules

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A Full Characterization of Equilibrium

A.1 Scale Independence for Firms' Problem

Below I give the proof to Proposition 1, restated here for convenience.

Proposition. *If $P_t(K, B)$ is homogeneous of degree 0 in (K, B), then the value function $V_t(K, B)$, optimal investment $I(K, B)$ and debt issuance $D(K, B)$ are homogeneous of degree 1 in (K, B). The default region can be defined in terms of b. Specifically, we have*

$$\begin{aligned}
V_t(K, B) &= v_t(b)K \\
I_t(K, B) &= \iota_t(b) K \\
D_t(K, B) &= d_t(b)K \\
R_t^{df} &= \{b \in \mathbb{R} | \max_{\iota, d} \Pi_t(\iota, d|1, b) < 0\}.
\end{aligned}$$

Proof. The proof amounts to deriving equity's HJB equation in terms of the leverage ratio b. Conjecture that the value function is in the form of $V_t(K, B) = v_t(b)K$, where $v_t(b)$ is the average value of equity per unit of capital, or *equity's average Q*. We have:

$$\begin{aligned}
\partial_K V(K, B) &= v(b) - bv'(b) \\
\partial_B V(K, B) &= v'(b) \\
\partial_{KK} V(K, B) &= v''(b)\frac{b^2}{K}
\end{aligned}$$

Plug it in (3.10), and use $P(\frac{B}{K}) \equiv P(K, B)$, using multiple dispatch[23], we have:

$$\begin{aligned}
(r_t + \zeta)v_t(b)K = \max_{\iota, d} \pi_t(\iota, d|b)K + \left(v_t(b) - bv_t'(b)\right)(\iota - \delta) K + v_t'(b)(d - \zeta b) K + \\
\frac{1}{2}v_t''(b)b^2\sigma^2 K + \xi \max\{1 - b, 0\} K + \partial_t v_t(b)K \\
0 \le \pi_t(\iota, d|b) \equiv (1 - \tau)\tilde{Z}_t + \tau(\delta + cb) - I - \varphi^K(\iota) - (\zeta + c)b + P_t(b) d - \varphi^B(d),
\end{aligned}$$

where $\iota \equiv \frac{I}{K}$ and $d \equiv \frac{D}{K}$, and $\pi_t(\iota, d|b) \equiv \Pi_t(\iota, d|1, b)$. As one-to-one mappings, these changes of variable do not affect the optimal control problem. Cancel out K and rearrange, we have the

[23]Multiple dispatch is a design of some programming languages (e.g. Julia) that the invocation (dispatch) of a particular version of the called function depends on its arguments. It is also commonly referred to as *an abuse of notation* in the literature.

HJB equation in terms of b:

$$
\begin{aligned}
(r_t + \zeta)v_t(b) = \max_{\iota,b} \pi(\iota, d|b) + (\iota - \delta) v_t(b) + \\
v_t'(b)(d - (\zeta + \iota - \delta)b) + \frac{1}{2}v_t''(b)\sigma^2 b^2 + \xi \max\{1 - b, 0\} + \partial_t v_t(b).
\end{aligned}
$$

The default region follows directly from the homogeneity of Π_t :

$$
\begin{aligned}
\mathbf{R}_t^{df} &= \left\{(K, B) \in \mathbb{R}^2 \middle| \max_{I,D} \Pi_t(I, D|K, B) < 0\right\} \\
&= \left\{(K, bK) \in \mathbb{R}^2 \middle| \max_{I,D} \pi_t\left(\frac{I}{K}, \frac{D}{K}|b\right) K < 0\right\} \\
R_t^{df} &= \{b \in \mathbb{R} | \max_{\iota,d} \pi_t(\iota, d|b) < 0\}.
\end{aligned}
$$

□

The proof to Corollary (1) immediately follows from the definition of R_t^{df}.

Corollary. *If $P_t(K, B)$ is homogeneous of degree 0 in (K, B) and weakly decreasing and continuous in leverage b, then there exists a threshold $\bar{b}_t$ such that the default region is $R_t^{df} = \{b \in \mathbb{R} | b > \bar{b}_t\}$.*

Proof. Plug in the quadratic functional forms of adjustment costs, the default region is given as

$$
\begin{aligned}
R_t^{df} &= \{b \in R | maxdiv_t(b) < 0\} \\
maxdiv_t(b) &\equiv (1 - \tau)\left(\tilde{Z}_t - cb - \delta\right) - \zeta b(1 - P_t(b)) + \frac{P_t^2(b)}{2\varphi_b} + \frac{1}{2\varphi_k}.
\end{aligned}
$$

For $b > 0$, the maximum dividend is strictly decreasing in b, provided a weakly decreasing P_t. We also have $maxdiv_t(0) > 0$, $\lim_{d \to \infty} maxdiv_t(b) < 0$, so there exist $\bar{b}_t$ such that $maxdiv_t(\bar{b}_t) = 0$, and $maxdiv_t(b) < 0$ for $b > \bar{b}_t$. □

A.2 Distribution

For grid efficiency, I index firms in the space (b, K) when analyzing the distribution. Denote $G_t(b, K)$ as distribution of firms at time t, and $g_t(d, K)$ as the corresponding density function. Given firms' optimal policy function $\mu_d(d)$, the distribution within the non-default region evolves according to the following Kolmogorov Forward Equation (KFE):

$$
\frac{dg_t(b, K)}{dt} = -\xi g_t(b, K) - \partial_d\left(g_t(b, K)\mu_t^b(b)\right) - \partial_K(g_t \iota_t(b)K) + \frac{1}{2}\partial_{bb}^2\left(g_t\sigma^2 b^2\right) + \frac{1}{2}\partial_{KK}\left(g_t\sigma^2 K^2\right) + \partial_{bK}^2\left(g_t\sigma^2 bK\right) \tag{A.1}
$$

The default boundary $(\bar{d}, K)$ is an absorbing-reinjection boundary, so that firms hitting the default boundary are immediately reinjected back to the state for entrants (b_0, K_0). In order to induce a stationary distribution, I also assume I also assume a lower bound on capital $\underline{\mathrm{K}}$ and an acquisition mechanism. Once a firm's capital is reduced to the lower bound, they will be acquired at the equity value and reinjected as a new entrant. The frequency of acquisition in the baseline calibration is almost trivial.

Notice that unlike the value equation, the density function does not preserve homogeneity because of the correlation between d and K, as shown in the cross-derivative term as well as the exit/entry process, so we need to solve the KFE on the two-dimensional space.

On a numerical note, the cross-derivative in the KFE also poses another challenge for the numerical algorithm. As well-known to the numerical PDE literature, a naive finite-difference scheme to approximate the cross derivative does not satisfy the monotonicity condition required for its stability. I use the local coordinate rotation method introduced by Ma and Forsyth (2016) to overcome this issue.[24]

A.3 Default Probabilities

I define $\tilde{Q}_t^T(b)$ as the default probability of firm with leverage b at time $t$before time T.$\tilde{Q}_t^T(b)$ can be expressed as the conditional expectation of a default indicator function:

$$\tilde{Q}_t^T(b) = \mathbb{E}_t\left[e^{-\zeta(T^{df}-t)}\mathbb{I}^T_{defaulted}|b_t = b\right].$$

where T^{df} is the stopping time when the firm first enters the default region. The term $e^{-\zeta(T^{df}-t)}$ reflects the probability of exogenous exits before T. The conditional expectation above can be computed recursively by the Feynman–Kac formula:

$$\partial_t\tilde{Q}_t^T(b) + \partial_b\tilde{Q}_t^T(b)\mu_t^b(b) + \frac{1}{2}\partial^2_{bb}v_t(b)\sigma^2 b^2 - \zeta\tilde{Q}_t(b) = 0,$$

with boundary conditions such that the default probability is 1 if the firm already entered the default region:

$$\tilde{Q}_t^T(\bar{b}_t) = 1\ \forall t \leq T.$$

A.4 Scale Independence for Bond Pricing

In this section I show the scale independence in the bond pricing function. Formally, suppose firms' policy functions are homogeneous of degree one, i.e., $I_t(K,B) = \iota_t(b)K$ and $D_t(K,B) =$

[24] Other methods are also available. See, e.g., d'Avernas and Vandeweyer (2021).

$d_t(b)K$, and default region is characterized by leverage b, $R_t^{df} = \{b \in R | maxdiv_t(b) < 0\}$, then bond pricing function is homogeneous of degree zero. This result, together with Proposition 1, establishes a symmetric equilibrium between the bond investor and firms.

To show the scale independence in price, I start from the definition of the price of firms

$$P_t(K,B) = \mathbb{E}_t\left[\int_t^{T^{df}} e^{-\int_t^h (r_\tau^c(b)+\zeta)d\tau}(c+\zeta)\,d\tau + e^{-\int_t^{T^{df}}(r_\tau^c(b)+\zeta)d\tau} P_{T^{df}}^{df}\left(\frac{B_{T^{df}}}{K_{T^{df}}}\right)|K_t = K, B_t = B\right].$$

As I focus on recursive equilibrium, this equation can be written as the HJB equation:

$$(r_\tau^c(b)+\zeta)P_t(K,B) - \partial_t P_t(K,B) = c + \zeta + \partial_K P_t(K,B)I + \partial_B P_t(K,B)(D-\zeta B) + \partial_{KK}^2 P_t(K,B)\sigma^2 K^2, \tag{A.2}$$

with boundary condition $P_t\left(K, \bar{b}_t K\right) = P_t^{df}\left(\bar{b}_t\right)$.

Guess $P(K,B) = P(\frac{B}{K})$. The boundary condition obviously satisfies this condition, as shown in Section A.5. Plug the guess in A.2 and cancels out K, we have:

$$\left(r_t^c(b)+\zeta\right)P_t(b) = c + \zeta + \partial_b P_t(b)\mu_t^b(b) + \frac{1}{2}\partial_{bb}^2 P_t(b) b^2\sigma^2 + \partial_t P_t(b). \tag{A.3}$$

Therefore, the bond price can indeed be solved as a function of b alone.

A.5 Defaulted bonds

In this section, I characterize defaulted bond price and discuss the market clearing condition. Defaulted bonds are modeled with sluggish payouts. A defaulted bond pays no coupon and awaits bankruptcy resolution, which happens at a Poisson rate ζ^{df}. The Poisson process captures the legal uncertainty around bankruptcy. Upon resolution, capital of the defaulted firm is liquidated at the recovery rate κ to pay back debt holders. The eventual payout per unit of bond is $\frac{\kappa K}{B} = \frac{\kappa}{b}$. Notice that when the liquidation happens instantaneously, the price of the defaulted bond is $\frac{\kappa}{b}$, constant across time. This feature is counterfactual as the literature has shown that during bad times the prices of defaulted bonds are particularly low (see, e.g., Jankowitsch et al., 2014). More importantly, bonds close to default may be priced even *lower* than defaulted bonds, as the former are subject to regulatory constraints while the latter is an immediate cash payout $\frac{\kappa}{b}$.

To price defaulted bonds, denote $P_t^{df}(b)$ as the price of the defaulted bond with leverage b. The homogeneity in (K,B) follows a similar argument as in Appendix A.4. [25] Its return

[25] As the default threshold $\bar{b}_t$ can be time-varying, the defaulted firms may also have different leverage b. However, their assets are frozen so their leverage is fixed at the time of default.

$dR_t^{df}(b) = \frac{\left(\frac{\kappa}{b} - P_t^{df}(b)\right) d\mathcal{J}_t^{df} + \partial_t P_t^{df}(b)}{P_t^{df}(b)}$, and the expected return is thus given as

$$r_t^{df}(b) = \frac{\zeta^{df}\left(\frac{\kappa}{b} - P_t^{df}(b)\right) + \frac{d}{dt} P_t^{df}(b)}{P_t^{df}(b)}. \tag{A.4}$$

Notice that once the path of return $r_t^{df}(b)$ is known, this equation gives an ODE of $P_t^{df}(b)$ as a function of time for each leverage ratio b. Together with terminal conditions for $P_t^{df}(b)$, these ODEs pin down the defaulted bond prices.

I assume the risk weight of defaulted bonds is constant, $rw(1)$, and they are treated as high-yield bonds subject to the HY constraint. The discount rate for defaulted bonds is therefore leverage-independent and given by:

$$r_t^{df} = r_t^G + \eta_t^{RW} rw(1) + \eta_t^{HY} \tag{A.5}$$

Combine (A.5) and (A.4), we can guess and verify that $P_t^{df}(b)$ can be written as $P_t^{df}(b) = \tilde{P}_t^{df} \frac{1}{b}$, where $\tilde{P}_t^{df}$ is the price of one unit of distressed capital, solved from the ODE below:

$$r_t^{df} \tilde{P}_t^{df} = \partial_t \tilde{P}_t^{df} + \zeta^{df}\left(\kappa - \tilde{P}_t^{df}\right).$$

To characterize the market clearing condition for defaulted bonds, observe that regardless of the leverage of the defaulted bonds, the total wealth invested in defaulted bonds is always equal to the total value of the distressed capital outstanding. Hence we only need to keep track of the defaulted capital, regardless of the respective leverage. Denote K_t^{df} the distressed capital stock. It follows the law of motion

$$dK_t^{df} = -\zeta^{df} K_t^{df} + \int K J_t(\bar{b}_t, K) dK,$$

where where $J_t(\bar{b}_t, K)$ is the density current flowing through the default threshold $\bar{b}_t$. Then the market clearing for defaulted bonds require

$$W_t^{df} = K_t^{df} \tilde{P}_t^{df}.$$